Proposed solution of prob. 1181

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In The Problem Department of Fall 2007 issue of this journal, readers were challenged to find a closed form expression for the trigonometric sum

$$\cos^{2n} 1^\circ + \cos^{2n} 2^\circ + \ldots + \cos^{2n} 89^\circ$$

where n is a positive integer. Here, the challenge is to find a closed form expression for the trigonometric sum

$$\cos^{2n+1} 1^\circ + \cos^{2n+1} 2^\circ + \ldots + \cos^{2n+1} 89^\circ$$

for $n \geq 0$

Answer: $\displaystyle \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \Big((-1)^{n-k} \cot \frac{2n+1-2k}{2} - 1 \Big)$

Proof We have (the angles in the trigonometric functions are expressed by degrees)

$$\sum_{p=1}^{89} \cos^{2n+1} p = \sum_{p=1}^{89} \frac{(e^{ip} + e^{-ip})^{2n+1}}{2^{2n+1}} = \sum_{p=1}^{89} \frac{1}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} e^{ip(2n+1-2k)}$$

$$\sum_{p=1}^{89} e^{ip(2n+1-2k)} = \frac{1 - e^{i90(2n+1-2k)}}{1 - e^{i(2n+1-2k)}} - 1 = \frac{e^{i(2n+1-2k)} - i(-1)^{n-k}}{1 - e^{i(2n+1-2k)}}$$

Thus

$$\sum_{p=1}^{89} \cos^{2n+1} p =$$

$$= \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \Big(\frac{e^{i(2n+1-2k)} - i(-1)^{n-k}}{1 - e^{i(2n+1-2k)}} - \frac{e^{-i(2n+1-2k)} + i(-1)^{n-k}}{1 - e^{-i(2n+1-2k)}} \Big) =$$

$$= \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{2\cos(2n+1-2k) - 2 + 2(-1)^{n-k} \sin(2n+1-2k)}{4 \sin^2 \frac{2n+1-2k}{2}} =$$

$$= \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \Big((-1)^{n-k} \cot \frac{2n+1-2k}{2} - 1 \Big)$$

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