Proposed solution to problem 1129 of
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Let R denote the real numbers and Q the rational numbers. A function f has a local minimum at the point x_0 if there exists an open neighborhood U of x_0 such that $f(x_0) \leq f(x)$ for all $x \in U$.

1. Find a non-constant function $f\colon R \to R$ such that f has a local minimum at every point.

2. Find a function $g\colon Q \to Q$ such that for each rational number r, there is neighborhood U of r such that $g(r) < g(x)$ for each $x \in U$.

1. Let $f\colon R \to R$ be defined by $f(x) = 1$ if $x < 0$ and $f(x) = 0$ for $x \geq 0$. f is non–constant and every point is a minimum.

2. $g\colon Q \to Q$, $g(p/q) = -1/q$, $g(0) = -1$ and (1 is the greatest common divisor of p and q: $(p|q) = 1$). For any p/q let's define a neighborhood by $\left|\frac{p}{q} - \frac{p'}{q'}\right| < \frac{1}{q^2}$, $p'/q' \neq p/q$. $\frac{1}{q^2} > \left|\frac{pq' - p'q}{qq'}\right| \geq \frac{1}{qq'}$ whence $q' \geq q + 1$. It follows $g(p'/q') = -1/q' > -1/q$. Of Course $x = 0$ is a minimum.

This function is not new. As far as I know it goes back to Riemann

- By the way $f\colon R \to R$, $f(x) = \begin{cases} -1/q, & x = p/q,\ (p|q) = 1 \\ 0 & \text{otherwise} \end{cases}$ has a dense set of minima of 0 Lebesgue measure. The set of minima of the everywhere dicontinuous Dirichlet function: $f(x) = 0$ $x \in Q$ and $f(x) = -1$ for $x \in R\backslash Q$ is of full measure.

In [2] the authors construct an example of *continuous* function $f\colon R \to R$ having a dense set of *proper* minima ($f(x_0) < f(x)$). This type of functions can be proved to be dense (residual) in the set of continuous functions $C([0,1])$ with the sup norm [3]. A considerably more difficult example of a *differentiable* function having a dense set of maxima and minima is constructed in [4], see also [5] p.141.

- As for point 1 we could have inserted "more steps": $f(x) = -k$ for $k \leq x < k+1$ whose graph is a "staircase". The steps can be made as close as we want but there does not exist a function satisfying 1 and not constant on every interval.We then prove the theorem

Theorem *There does not exist a function $f\colon R \to R$, having a minimum at each point and not constant on every open neighborhood*

The initial step of the proof is the following interesting lemma (proved as early as 1900 [1])

Lemma *The set of the ordinates of maxima or minima is a countable set for any function $f\colon R \to R$,*

Proof of the theorem Let be $f\colon R \to R$ and $B = f(R)$. By hypotheses each point of B is a minimum and the lemma implies the countability of B: $B = \bigcup_{k=1}^{\infty} y_k$. Let's define $A_k \doteq f^{-1}(y_k)$ so that $R = \bigcup_{k=1}^{\infty} A_k$, $(A^{(k_0)} \doteq \bigcup_{k=1, k\neq k_0}^{\infty} A_k)$.

By the Baire–category theorem applied on R which is complete, we have $\overset{\circ}{\overline{A}}_k \neq \emptyset$ for at least a k, say $k = k_0$. Hence there exists an open set, say C, such that for any $p \in C$, $A_{k_0} \cap U_p \neq \emptyset$ for any open neighborhood $U_p \ni p$. There are two possibilities:

i) If $U_p \cap A^{(k_0)} = \emptyset$ for a pair p–U_p, then $U_p \subset A_{k_0}$ and this would imply f constant and equal to y_{k_0} in U_p hence the thesis.

ii) If $U_p \cap A^{(k_0)} \neq \emptyset$ for any $p \in C$ and for any $U_p \ni p$, then $A^{(k_0)}$ would be dense in C. In this case the density of $A^{(k_0)}$ and A_{k_0} contradicts the fact that each point must be a minimum.

• The Lemma is false if one wants countable the set of the inflection points and in fact a C^1 counterexample is easily constructed. Let be $F \subset [0,1]$ the Cantor–ternary–set and for $x \in [0,1]$, $\rho(x,F)$ is the distance between x and F. Let be $h(x) \doteq \int_0^x \rho(t,F)dt$.

1) h is differentiable and $h'(x) = \rho(x,F)$ being $\rho(x,F)$ continuous , 2) $h'(x) \geq 0$ being $\rho(x,F) \geq 0$. The derivative is zero if and only if $x \in F$ being F a closed set 3) h is injective. In fact $\int_x^{x'} \rho(t,F)dt > 0$ if $x < x'$ because F is completely disconnected, (F does not contain any interval), 4) the points of zero derivative are uncountable being F uncountable (as well known).

The Cantor–ternary–set has zero Lebesgue measure but this is inessential. We could have taken a Cantor set of positive measure.

References

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