same length. (It is in fact a rhombus.) It follows that $PH = AO = R$. Thus as claimed H lies on $\mathcal{C}'$.

Editorial comment. The Blundon result from E2282 may be strengthened in an interesting way due to Francisco Bellot Rosado (Spain), who submitted it to this MONTHLY in 1998 as part of a solution to Problem 10547: Let G denote the centroid of the triangle. The incenter I always lies inside the circle whose diameter is GH, because the angle GIH is always obtuse. Since the perpendicular bisector λ of the Euler segment OH divides the circle of Bellot Rosado into a larger and a smaller piece, I is (1) in the larger piece, (2) on line λ, or (3) in the smaller piece, according as the middle-sized angle of ABC is (1) greater than, (2) equal to, or (3) less than $60°$.

Also solved by M. Bataille (France), R. Chapman (U. K.), C. Curtis, Y. Dumont (France), D. Fleischman, V. V. Garcia (Spain), D. Grinberg, J.-P. Grivaux (France), E. Hysnelaj (Australia) & E. Bojaxhiu (Albania), O. Kouba (Syria), J. H. Lindsey II, J. Minkus, R. Stong, M. Tetiva (Romania), D. Vacaru (Romania), Z. Vörös (Hungary), M. Vowe (Switzerland), J. B. Zacharias & K. Greeson, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

An Alternating Series

11409 [2009, 83]. *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.* For positive real α and β, let

$$S(\alpha, \beta, N) = \sum_{n=2}^{N} n \log(n)(-1)^n \prod_{k=2}^{n} \frac{\alpha + k \log k}{\beta + (k+1)\log(k+1)}.$$

Show that if $\beta > \alpha$, then $\lim_{N\to\infty} S(\alpha, \beta, N)$ exists.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. Let $\omega_k = k \log k$. Write

$$a_n = \omega_n \prod_{k=2}^{n} \frac{\alpha + \omega_k}{\beta + \omega_{k+1}} = b_n \prod_{k=3}^{n} \left(1 - \frac{\beta - \alpha}{\beta + \omega_k}\right), \quad \text{where} \quad b_n = \frac{(\alpha + \omega_2)\omega_n}{\beta + \omega_{n+1}}, \quad (1)$$

and suppose $\beta > \alpha$. We will prove that

$$\sum_{n=2}^{\infty} (-1)^n a_n \quad \text{converges},$$

so $\lim_{N\to\infty} S(\alpha, \beta, N)$ exists. By the alternating series test of Leibniz, and noting $a_n > 0$, it suffices to prove

(i) $a_{n+1}/a_n < 1$ for all sufficiently large n, and

(ii) $a_n \to 0$ as $n \to \infty$.

(i) From the definition of a_n in (1),

$$\frac{a_{n+1}}{a_n} = \frac{\omega_{n+1}(\alpha + \omega_{n+1})}{\omega_n(\beta + \omega_{n+2})},$$

so $a_{n+1}/a_n < 1$ is equivalent to $\omega_{n+1}\,\alpha + (\omega_{n+1}^2 - \omega_n\,\omega_{n+2}) < \omega_n\,\beta$. Calculation shows $\omega_{n+1}^2 - \omega_n\,\omega_{n+2} = (\log n)^2 + \log n + 1 + o(1)$. Because $\beta > \alpha$ and $\omega_{n+1} \sim \omega_n = n \log n$, the required result follows.

(ii) Because $\lim_{n\to\infty} b_n$ exists, to show $\lim_{n\to\infty} a_n = 0$ it suffices to show that the infinite product

$$\prod_{k=3}^{\infty} \left(1 - \frac{\beta - \alpha}{\beta + \omega_k}\right) \quad (2)$$

diverges to zero. Recall that if $0 < c_k < 1$ for all k and $\sum_{k=1}^{\infty} c_k$ diverges, then $\prod_{k=1}^{\infty}(1 - c_k)$ diverges to 0. In the present case, the divergence of

$$\sum_{k=3}^{\infty} \frac{1}{\omega_k} = \sum_{k=3}^{\infty} \frac{1}{k \log k}$$

shows that the infinite product in (2) diverges to 0. (That the sum diverges is well known, as it follows from the integral test or Cauchy condensation test.)

Also solved by S. Amghibech (Canada), P. Bracken, R. Chapman (U. K.), P. P. Dályay (Hungary), D. Grinberg, J. Grivaux (France), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

A Fix for a Triangle Inequality

11413 [2009, 179]. *Proposed by Mihály Bencze, Brasov, Romania.* Let θ_i for $1 \le i \le 5$ be nonnegative, with $\sum_1^3 \theta_i = \pi$, $\theta_4 = \theta_1$, and $\theta_5 = \theta_2$. Let $S = \sum_{i=1}^3 \sin\theta_i$. Show that

$$S \le \frac{3\sqrt{3}}{2} - 4 \max_{1 \le i \le 3} \left(\sin^2\left(\frac{1}{(4)}(\theta_i - \theta_{i+1})\right) \cos\left(\frac{1}{2}\theta_{i+2}\right) + \sqrt{3}\sin^2\left(\frac{1}{12}(\pi - 3\theta_{i+2})\right)\right).$$

Solution by Richard Stong, San Diego, CA. (The originally published statement had a misprint, with "2" where "(4)" now stands.) If $A, B, C \ge 0$ with $A + B + C = \pi$, then

$$S = \sin A + \sin B + \sin C = 4\cos(A/2)\cos(B/2)\cos(C/2).$$

Hence

$$\begin{aligned} S + 4\sin^2((A - B)/4)\cos(C/2) &= 4\cos^2((A + B)/4)\cos(C/2) \\ &= 4\cos^2(\pi - C)/4)\cos(C/2). \end{aligned}$$

Applying the identity

$$4\cos(x + 2y)\cos^2(x - y) + 8\sin^2 y \cos x = 4\cos^3 x - 4\sin^2 y \cos(x - 2y)$$

with $x = \pi/6$ and $y = (\pi - 3C)/12$, we have

$$4\cos\frac{C}{2}\cos^2\frac{\pi - C}{4} + 4\sqrt{3}\sin^2\frac{\pi - 3C}{12} = \frac{3\sqrt{3}}{2} - 4\sin^2\frac{\pi - 3C}{12}\cos\frac{2\pi - 3C}{6}$$

or, combined with the above,

$$S + 4\sin^2\frac{A - B}{4}\cos\frac{C}{2} + 4\sqrt{3}\sin^2\frac{\pi - 3C}{12} = \frac{3\sqrt{3}}{2} - 4\sin^2\frac{\pi - 3C}{12}\cos\frac{2\pi - 3C}{6}.$$

Since $0 \le C \le \pi$, the last cosine is nonnegative, and hence

$$S + 4\sin^2\frac{A - B}{4}\cos\frac{C}{2} + 4\sqrt{3}\sin^2\frac{\pi - 3C}{12} \le \frac{3\sqrt{3}}{2}.$$

Apply this result three times, taking (A, B, C) to be $(\theta_1, \theta_2, \theta_3)$, then $(\theta_2, \theta_3, \theta_1)$, and finally $(\theta_3, \theta_1, \theta_2)$, to obtain the desired result.

Editorial comment. Some solvers corrected the problem by showing that it holds as originally printed but with the inequality reversed.