PROBLEMS

11159. *Proposed by George Lamb, Tucson, AZ.* For $|a| < \pi/2$, evaluate in closed form

$$I(a) = \int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos\psi \, d\psi \, d\varphi}{\cos(a\cos\psi\cos\varphi)}.$$

11160. *Proposed by Marc Chamberland, Grinnell College, Grinnell, IA.* Let a, b, and ϕ be real and x, y, and z be rational with $xyz = 0$. Given that $x = a + b\cos\phi$, $y = a + b\cos(\phi - 2\pi/3)$, $z = a + b\cos(\phi + 2\pi/3)$, and $27ab^2 = 4$, prove that (x, y, z) is a permutation of $(1, 0, 0)$.

11161. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Show that for all integers $n \geq 3$ the number of *compositions* of n into relatively prime parts is a multiple of 3. (A composition of n into k parts is a list of k positive integers that sum to n. Thus, there are six compositions of 4 into relatively prime parts: $(3, 1)$, $(1, 3)$, $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$, and $(1, 1, 1, 1)$.)

11162. *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.*
(a) Show that if c is a real number less than 2 then $\sum_{k=1}^{\infty} k^{-c-\sin k}$ diverges.
(b) Determine whether $\sum_{k=1}^{\infty} k^{-1-|\sin k|}$ converges.

11163. *Proposed by Michel Bataille, Rouen, France.* Let c and n be positive integers with $n > c^2$. Let $q_{n,c}$ denote the number of quadrilaterals with vertices at integer lattice points and sides tangent to the ellipse with equation $x^2/n + y^2/(n - c^2) = 1$.
(a) For which c and n is $q_{n,c}$ positive?
(b) Show that, for $c \geq 1$, $\sup_{n>c^2} q_{n,c} = \infty$.
(c)* Is $q_{n,c}$ finite for all n and c?

11164. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.* Show that if n is a positive integer, then

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sum_{1 \leq i \leq j \leq k} \frac{1}{ij} = \frac{1}{n^2}.$$

11165. *Proposed by Yogesh More, University of Michigan, Ann Arbor, MI.* Let C_k be the kth Catalan number, $\frac{1}{k+1}\binom{2k}{k}$. Prove that, for each positive integer n, $\sum_1^n C_k \equiv 1 \pmod 3$ if and only if the base 3 expansion of $n + 1$ contains the digit 2. Find similar characterizations for the other two cases, in which the sum is congruent to 0 or 2 modulo 3.