Problem Section

Editors

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This section features problems for students at the undergraduate and (challenging) high school levels. Problems designated by "S" are especially well-suited for students at all levels. All problems and/or solutions should be submitted to Derek Smith, Mathematics Department, Lafayette College, Easton, PA 18042. Electronic submissions may also be sent to `smithder@lafayette.edu`. Please include your name, email address, school affiliation, and indicate if you are a student. Electronic submissions are preferred.

Proposals

To be considered for publication, solutions to the following problems should be received by June 10, 2008.

Problem 213. *Proposed by Ben Bielefeld.* Determine the number of different ways that a $5 \times 5 \times 5$ cube can be constructed using five $1 \times 1 \times 1$ blocks, six $1 \times 2 \times 4$ blocks, and six $2 \times 2 \times 3$ blocks.

Problem 214. *Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma.* Let $[a]$ represent the integer part of a real number a; for example, $[\pi] = 3$. Evaluate

$$\int_0^1 \int_0^{1-y} \frac{1}{\left[\frac{x}{y}\right]+1}\, dx\, dy.$$

Problem 215. *Proposed by Tom Yuster, Lafayette College.* You need to access a file in a directory on a server. You know that the directory contains only that file; moreover, you know the name of the file is `file`*xyz*, where each of x, y, and z is a digit in the range $0, 1, \ldots, 9$. However, you don't recall the value of any of these three digits, and the server doesn't let you list the contents of the directory: you have to request a single filename at a time.

You do recall that the server uses a specific single-digit correction routine, so that `file`*xyz* will be found if you request any of the filenames `file`*xy*, `file`*yz*, or `file`*xz*. Suppose you hunt for the file by typing in requests for files of the form `file`*ab*, where a and b are digits. What is the minimum number of requests you have to make to be assured of finding the file?

Problem 216. *Proposed by students Ashutosh Priyadarshy and Rana Singh, Governor's School, Lynchburg College.* In 1615, Galileo observed that

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots$$

Motivated by this observation, show that the sequence

$$\frac{1}{(1+a)^k}, \frac{1+(1+a)^k}{(1+2a)^k+(1+3a)^k}, \frac{1+(1+a)^k+(1+2a)^k}{(1+3a)^k+(1+4a)^k+(1+5a)^k}, \ldots$$

converges to $1/(2^{k+1}-1)$ for all natural numbers a and k.

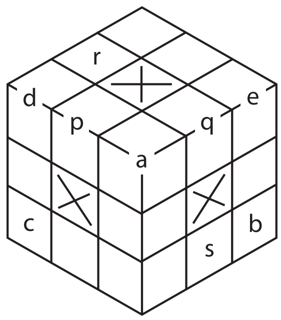
Solutions:

Problem 209. Tic-Tac-Draw? *Proposed by Ethan Berkove.* The following generalization of tic-tac-toe is played on an $n \times n \times n$ cubical grid. Two players alternately place (3-dimensional) Xs and Os in the grid until either (a) one of the two players completes a straight line of n of the same symbol or (b) there is a "draw" when the board is filled and no such line has been created.

For the case $n = 2$, it is not possible for the players to conspire to play to a draw since the first player must win on her second move. For each of the cases $n = 3$ and $n = 4$, is it possible for the players to draw?

Solution for $n = 3$ by the proposer. Here is a proof that two players cannot conspire to draw. Assume that the first player uses Xs and the second player uses Os, so that a total of 14 Xs and 13 Os would be used to fill the cells of a drawn game. Notice that if an X is placed in the center cell of the cube, each of the 13 lines through the center would have to have an O at one end and an X at the other end; on the other hand, if an O is placed in the center cell, the same is true of 12 of the 13 lines through the center, while the remaining line must have Xs at both ends. In either case, there must exist a corner of the cube for which the view of its three adjacent faces looks like the figure at the top of the following page, with Xs in the center of the faces.

Call two cells at the ends of lines through the center of the cube "opposite" cells. Can an X be placed in the corner cell (a)? No, for then the cells (b) and (c) must get Os, which means their opposite cells (d) and (e) get Xs, for three Xs in a row across the top face of the cube. Can an X be placed on the edge cell (q)? No, for then the cells (r) and (s) must both get Os, an impossibility because they are opposite cells. Similar reasoning applies to the edge cell (p), so that without loss of generality there is a corner of the cube as shown above with Os at (a), (p), and (q). But now any symbol choices for cells (d) and (e) must complete a line of three of the same symbol.



Solution for $n = 4$ by student Damien Mondragon, University of the Pacific. A picture is all that is needed to prove that a draw is possible for $n = 4$. The 4 × 4 grids displayed below are to be stacked atop each other in the order A, B, C, D to make a 4 × 4 × 4 cube.

O	X	O	X
X	O	X	O
X	O	X	O
O	X	O	X

A

X	O	X	O
O	X	O	X
O	X	O	X
X	O	X	O

B

O	X	O	X
X	O	X	O
X	O	X	O
O	X	O	X

C

O	X	O	X
X	O	X	O
X	O	X	O
O	X	O	X

D

Solutions for both cases were given independently by students Michael Eldredge, Andrzej Grzesik, and Damien Mondragon, each of whom (for $n = 3$) essentially used the fact that there are, up to symmetries, just three different ways to create a draw on a standard 3 × 3 tic-tac-toe board. Trying each one of these three possibilities as the bottom face of the cube, and considering whether the central cell is an X or an O, rather quickly leads to the conclusion that a line of 3 of the same symbol must always be formed. Students Dominic Marasco, Aimee Archbold, and Cody Hubert, with Dr. Mark Sand, provided a solution for the $n = 4$ case only. There was one incorrect solution.

Credit for both cases should also be given to students Jennifer Griffeth and Katie Zirschky, who give a reference to the 1978 *Mathematics Magazine* article "Tic-Tac-Toe in n dimensions" by Jerome Paul, where these and other tic-tac-toe extensions are discussed. Paul calls configurations like the 4 × 4 × 4 labeling shown above "tie partitions." It should be noted that the existence of tie partitions does not imply that a game played by two rational players will necessarily be a draw. On the contrary, as reported by Martin Gardner in his January 1979 *Scientific American* column, Oren Patashnik of Bell Laboratories wrote a computer program in 1977 (requiring 1,500 hours of computing time!) to first verify that tic-tac-toe on a 4 × 4 × 4 grid is a first player win.

Problem 210. Small Cliques. *Proposed by Alex Wilce*. Let S be a set of points in the unit square

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \le x, y \le 1\}$$

containing no diagonal point (x, x). Show that if S is closed then S contains no infinite clique.

(There are many ways to define a *closed* set S: for instance, require that each convergent sequence of points in S has a limit that is also in S. Here, a *clique* is the set of all ordered pairs (x, y) of distinct points $x, y \in T$, where T is a subset of [0, 1].)

Solution by student Ryan Eberhart, Rochester Institute of Technology. This is a proof by contraposition. Suppose S contained an infinite clique, consisting of the set of all ordered pairs (x, y) of distinct points $x, y \in T$. This means that T itself must be infinite, so there exists an infinite sequence $\{t_i\}$ of distinct points in T. Since $\{t_i\}$ is a bounded sequence in the closed interval [0, 1], by the Bolzano-Weierstrass theorem there exists some subsequence $\{t_{n_i}\}$ of $\{t_i\}$ and $t \in [0, 1]$ such that $\{t_{n_i}\}$ converges to t. Thus, the sequence $(t_{n_i}, t_{n_{i+1}})$ of points in S converges to (t, t), a diagonal point that would be in S if S were closed.

Solutions given by Andrzej Grzesik (student), Damien Mondragon (student), Michael Woltermann, the Northwestern University Math Problem Solving Group, and the Skidmore College Problem Group were essentially the same as Ryan's solution. There were two incorrect solutions. Damien noticed that the Bolzano-Weierstrass theorem made an appearance on the page of the September issue immediately before Problem 210 was posed. A coincidence? (Yes.)

The problem proposer offers an alternate proof, one occurring more naturally in his research in topology (where a more general version of the problem replaces the interval [0, 1] with any compact Hausdorff space). In the alternate proof, assuming S is closed and contains no diagonal point, each point $x \in [0,1]$ is contained in an open interval I_x such that the open square $I_x \times I_x$ does not intersect S. The collection of open sets $\{I_x\}$ covers [0,1], and therefore a *finite* subcollection of these sets will cover [0,1] as well... which might give you more of a hint than you need to finish this proof!

Problem 211. Quantum Pin Cushion. *Proposed by an editor.* Here is a game of solitaire to play with a finite set S of lines through the origin in $\mathbb{R}^3$. Add new lines to S according to the following rule: if S contains two lines that are orthogonal to each other, the line orthogonal to both of these lines is added to S if that line is not already contained in S. For example, verify that starting with 3 lines in S can lead to 0, 1, or 2 additional lines.

Suppose that a finite set S initially contains only lines that are the spans of integral vectors in $\mathbb{R}^3$; that is, each line initially in S consists of all real multiples of a vector (x, y, z) with x, y, z integers. Must the game terminate, or can the size of S increase without bound?

Solution outline by the proposer. This was certainly the most difficult of the problems posed in the September column; there were two incorrect solutions. One proof that the game must terminate is outlined below. The most general version of this problem, where the initial set of lines is not restricted to consist only of spans of particular types of vectors, is unsolved.

If two lines in S are the spans of orthogonal vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, then the square of the i^{th} coordinate of their cross-product $\vec{v}\times\vec{w}$ is given by

$$(\vec{v}\times\vec{w})_i^2 = [\vec{v}][\vec{w}] - v_i^2[\vec{w}] - w_i^2[\vec{v}],$$

where $[\vec{a}] = a_1^2 + a_2^2 + a_3^2$ gives the squared length of a vector $\vec{a}$. Notice that adding the three instances of the just displayed equation for $i = 1, 2, 3$ gives a more familiar cross-product identity for orthogonal vectors,

$$[\vec{v}\times\vec{w}] = [\vec{v}][\vec{w}].$$

Now suppose that S is a finite set of lines, with each line in S spanned by a vector $\vec{u}$ with integral coordinates. Each such $\vec{u}$ can be assumed to be "primitive," meaning that its three coordinates have no common positive integer factor greater than 1, since dividing all of the coordinates by such a factor would still give the same line in S. This means that the finite set $\{[\vec{u}]\}$ consisting of all squared lengths of primitive spanning vectors of S determines a well-defined finite set of prime numbers $p_1, \ldots, p_n$ which occur as factors of some $[\vec{u}]$, where the highest power of p_i occurring among the $[\vec{u}]$ will be denoted by $p_i^{a_i}$. Using these facts and the two displayed equations above, all that remains is to show that any line added to S is spanned by a primitive vector whose squared length is also a product of the primes $p_1, \ldots, p_n$, with the highest power of any p_i being $p_i^{a_i}$, there being only finitely-many such vectors. (Whew!)

If someone can find a simpler proof than the one just outlined, please send it in.

Problem 212. Three-Year Integral. *Proposed by Mohammad Azarian.* Show that

$$\int_0^1 \frac{4x^3(1+x^{4(2006)})}{(1+x^4)^{2008}}\,dx = \frac{1}{2007}.$$

Solution by student Phan Le, Alfred University. First break the integral into two parts as

$$\int_0^1 \frac{4x^3}{(1+x^4)^{2008}}\,dx + \int_0^1 \frac{4x^3\cdot x^{4(2006)}}{(1+x^4)^{2008}}\,dx.$$

With the substitution $u = 1 + x^4$ the first part becomes

$$\int_1^2 \frac{1}{u^{2008}}\,du = \frac{1}{2007}(1-2^{-2007}),$$

while the second part becomes

$$\int_1^2 \frac{(u-1)^{2006}}{u^{2008}}\,du = \int_1^2 \frac{(1-1/u)^{2006}}{u^2}\,du.$$

With the additional substitution $v = 1 - 1/u$ the last integral becomes

$$\int_0^{1/2} v^{2006}\,dv = \frac{1}{2007}(2^{-2007}),$$

so the original integral is equal to

$$\frac{1}{2007}(1-2^{-2007}) + \frac{1}{2007}(2^{-2007}) = \frac{1}{2007}.$$

This problem was also solved by students Rene Ardila, Ana Burgers, Jacky Chen, Kyle Claassen, Jacqueline Diaz, Rebecca Greenwood, Wang Haining, Michael Janas, Wen Hao Liu, Sadiq Mohammed, Damien Mondragon, Norma Morris (with Farley Mawyer), Tim Treloar, Sandi Xhumari, Shuo Yang, Qiaochu Yuan, Katie Zirschky, the Fort Hays State University Problem Solving Group, Kappa Mu Epsilon at Northeastern State University (OK), the Northwestern University Math Problem Solving Group, and the Tulane University Math Club. Other solutions came in from James Camacho Jr., Minh Can, John Chase and Matt Davis, Michael Faleski, Allan Fuller and Satyajit Karmakar, Jerome Heaven, James R. Hochschild, Ronald Kopas, Eli Maor, Samih Obaid, Paolo Perfetti, Henry Ricardo, Fary Sami, Rachael Stedman and Carl Tappan, Zengxiang Tong, John Travis, Michael Woltermann, and the proposer. Many of the solutions essentially used the substitutions given above, while others used binomial expansions or trigonometric substitutions. Several generalizations of the integral were stated and proved. There was one incorrect solution.