PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before October 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11579. *Proposed by Hallard Croft, University of Cambridge, Cambridge, U. K., and Sateesh Mane, Convergent Computing, Shoreham, NY.* Let m and n be distinct integers, with $m, n \geq 3$. Let B be a fixed regular n-gon, and let A be the largest regular m-gon that does not extend beyond B. Let $d = \gcd(m, n)$, and assume $d > 1$. Show that:
(**a**) A and B are concentric.
(**b**) If $m \mid n$, then A and B have m points of contact, consisting of all the vertices of A.
(**c**) If $m \nmid n$ and $n \nmid m$, then A and B have $2d$ points of contact.
(**d**) A and B share exactly d common axes of symmetry.

11580. *Proposed by David Alfaya Sánchez, Universidad Autónoma de Madrid, Madrid, Spain, and José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.* For $n \geq 2$, let $a_1, \dots, a_n$ be positive numbers that sum to 1, let $E = \{1, \dots, n\}$, and let $F = \{(i, j) \in E \times E\colon\ i < j\}$. Prove that

$$\sum_{(i,j)\in F} \frac{(a_i - a_j)^2 + 2a_i a_j (1 - a_i)(1 - a_j)}{(1 - a_i)^2 (1 - a_j)^2} + \sum_{i \in E} \frac{(n+1)a_i^2 + n a_i}{(1 - a_i)^2} \geq \frac{n^2(n+2)}{(n-1)^2}.$$

11581. *Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam.* Let f be a continuous, nonconstant function from $[0, 1]$ to $\mathbb{R}$ such that $\int_0^1 f(x)\,dx = 0$. Also, let $m = \min_{0\leq x\leq 1} f(x)$ and $M = \max_{0\leq x\leq 1} f(x)$. Prove that

$$\left|\int_0^1 x f(x)\,dx\right| \leq \frac{-mM}{2(M-m)}.$$

11582. *Proposed by Aleksandar Ilić, University of Niš, Serbia.* Let n be a positive integer, and consider the set S_n of all numbers that can be written in the form $\sum_{i=2}^{k} a_{i-1}a_i$ with $a_1, \dots, a_k$ being positive integers that sum to n. Find S_n.

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11583. *Proposed by David Beckwith, Sag Harbor, NY.* The instructions for a magic trick are as follows: Pick a positive integer n. Next, list all partitions of n as nondecreasing strings—for instance, with $n = 3$, the list is $\{111, 12, 3\}$. Count 1 point for the string (n). For the string $\lambda_1 \cdots \lambda_k$ with $k > 1$, count $\prod_{j=1}^{k-1} \binom{\lambda_{j+1}}{\lambda_j}$ points. Add up your points, take the log base 2 of that, and add 1. Voilà! n. Explain.

11584. *Proposed by Raymond Mortini and Jérôme Noël, Université Paul Verlaine, Metz, France.* Let $\langle a_j \rangle$ be a sequence of nonzero complex numbers inside the unit circle such that $\prod_{k=1}^{\infty} |a_k|$ converges. Prove that

$$\left| \sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{a_j} \right| \le \frac{1 - \prod_{j=1}^{\infty} |a_j|^2}{\prod_{j=1}^{\infty} |a_j|}.$$

11585. *Proposed by Bruce Burdick, Roger Williams University, Bristol, RI.* Show that

$$\sum_{k=3}^{\infty} \frac{1}{k} \left(\sum_{m=1}^{k-2} \zeta(k-m)\zeta(m+1) - k \right) = 3 + \gamma^2 + 2\gamma_1 - \frac{\pi^2}{3}.$$

Here, ζ denotes the Riemann zeta function, γ is the Euler-Mascheroni constant, given by $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} 1/k - \log(n) \right)$, and γ_1 is the first Stieltjes constant, given by $\gamma_1 = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\log k}{k} - \frac{1}{2}(\log n)^2 \right)$.

SOLUTIONS

Extrema On the Edge

11449 [2009, 647]. *Proposed by Michel Bataille, Rouen, France.* (corrected) Find the maximum and minimum values of

$$\frac{(a^3+b^3+c^3)^2}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}$$

given that $a + b \ge c > 0$, $b + c \ge a > 0$, and $c + a \ge b > 0$.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO. Let F be the expression to be maximized. The maximum of F in the feasible region is 2, attained when $a = b = 1$ and $c = 2$, as well as at permutations and scalings of this.

Let $H = 2(b^2 + c^2)(c^2 + a^2)(a^2 + b^2) - (a^3 + b^3 + c^3)^2$. Since $F \le 2$ is equivalent to $H \ge 0$, we prove the latter. By symmetry, we may assume that $a \le b \le c$. By homogeneity, we may take $a = 1$. Hence, we can write $b = 1 + x$ and $c = 1 + x + y$ with $x, y \ge 0$. Since $a + b \ge c$, we have $y \le 1$. Expanding H as a polynomial in x with coefficients that are polynomials in y gives the following expansion:

$$\begin{aligned} H = {} & x^4[1 + 7(1+y)(1-y)] + 2x^3[1 + (1-y)(7y^2 + 21y + 13)] \\ & + x^2[1 + (1+y)(1-y)(13y^2 + 42y + 39)] \\ & + 2x(1+y)(1-y)(3y+7)(y^2+2y+2) + (1+y)^2(1-y)(y^3 + 5y^2 + 7y + 7), \end{aligned}$$

which is evidently nonnegative. It is 0 if and only if $x = 0$ and $y = 1$. This corresponds to $(a, b, c) = (1, 1, 2)$.

Also solved by R. Agnew, A. Alt, M. Ashbaugh, R. Bagby, D. Beckwith, H. Caerols & R. Pellicer (Chile), R. Chapman (U. K.), H. Chen, C. Curtis,P. P. Dályay (Hungary), Y. Dumont (France), J. Fabrykowski and T. Smotzer, S. Falcón and Á. Plaza (Spain), D. Fleischman, J.-P. Grivaux (France), E. A. Herman, F. Holland (Ireland), T. Konstantopoulos (U. K.), O. Kouba (Syria), A. Lenskold, J. H. Lindsey II, B. Mulansky (Germany), P. Perfetti (Italy), C. R. Pranesachar (India), N. C. Singer, R. Stong, T. Tam, R. Tauraso (Italy), M. Tetiva (Romania), D. Tyler, E. I. Verriest, Z. Vörös (Hungary), S. Wagon, G. D. White, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Editorial comment. Two versions of this problem appeared; the first was not what the proposer intended. The treatment of the upper bound given in the March issue of this column (p. 278) fails as a solution to the corrected version. The maximum of F in the closure of the feasible region is attained not only at a corner, which is off-limits, but also at the other boundary points noted. The solver list here includes those who had supplied solutions under a new deadline. The editors regret the confusion.

Hexagon Inscribed in Circle

11470 [2009, 491]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let $ABCDEF$ be a hexagon inscribed in a circle. Let M, N, and P be the midpoints of the line segments BC, DE, and FA, respectively, and similarly let Q, R, and S be the midpoints of AD, BE, and CF. Show that if both MNP and QRS are equilateral, then the segments AB, CD, and EF have equal lengths.

Solution by Oliver Geupel, Brühl, NRW, Germany. Let the circle be the unit circle in the complex plane, and let $a, b, c, \ldots$ be the complex numbers corresponding to $A, B, C, \ldots$. Thus $2m = b + c$, $2n = d + e$, $2p = f + a$, $2q = a + d$, $2r = b + e$, and $2s = c + f$. Write $\epsilon = \exp(2\pi i/3)$. It is well known (for example: T. Andreescu and T. Andrica, *Complex Numbers from A to Z*, Birkhäuser, Boston, 2006, pp. 70*ff.*, Proposition (3.4)1) that a triangle UVW is equilateral if and only if $u + \epsilon v + \epsilon^2 w = 0$ or $u + \epsilon w + \epsilon^2 v = 0$, depending on the orientation of $\triangle UVW$. Without loss of generality, we may assume that $\triangle MNP$ is oriented so that $m + \epsilon n + \epsilon^2 p = 0$. Hence

$$(b + c) + \epsilon(d + e) + \epsilon^2(f + a) = 0. \tag{1}$$

We consider two cases, depending on the orientation of $\triangle QRS$.

Case 1: $\triangle MNP$ and $\triangle QRS$ have opposite orientation. In this case

$$(a + d) + \epsilon(c + f) + \epsilon^2(b + e) = 0. \tag{2}$$

Multiplying (1) by $\frac{-1-\epsilon+\epsilon^2}{2(\epsilon-1)}$, multiplying (2) by $\frac{-1+\epsilon+\epsilon^2}{2(\epsilon-1)}$, and adding, we obtain $a + \epsilon c + \epsilon^2 e = 0$. Multiplying (1) by $\frac{-1+\epsilon+\epsilon^2}{2(\epsilon-1)}$, multiplying (2) by $\frac{-1-\epsilon+\epsilon^2}{2(\epsilon-1)}$, and adding, we obtain $b + \epsilon d + \epsilon^2 f = 0$. Thus $\triangle ACE$ and $\triangle BDF$ are equilateral, which implies $AB = CD = EF$.

Case 2: $\triangle QRS$ has the same orientation as $\triangle MNP$. Now

$$(b + e) + \epsilon(c + f) + \epsilon^2(a + d) = 0. \tag{3}$$

Multiplying (1) by $\frac{1}{1-\epsilon}$, multiplying (3) by $-\frac{1}{1-\epsilon}$, and adding, we obtain $c - e = \epsilon(f - d)$. Therefore $CE = DF$, so $CD = EF$. Multiplying (1) by $\frac{\epsilon^2}{1-\epsilon}$, multiplying (3) by $-\frac{1}{1-\epsilon}$, and adding, we obtain $e - a = \epsilon(f - b)$. Therefore $EA = FB$, so $EF = AB$.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), M. Garner, M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), S. W. Kim (Korea), O. Kouba (Syria), O. P. Lossers (Netherlands), M. A. Prasad (India), R. Stong, S. Tonegawa & F. Vafa, and the proposer.

Product of Derivatives

11472 [2009, 941]. *Proposed by Mahdi Makhul, Shahrood University of Technology, Shahrood, Iran.* Let t be a nonnegative integer, and let f be a $(4t+3)$-times continuously differentiable function on $\mathbb{R}$. Show that there is a number a such that at $x = a$,

$$\prod_{k=0}^{4t+3} \frac{d^k f(x)}{dx^k} \geq 0.$$

Solution by Robin Chapman, University of Bristol, Bristol, England, U. K. We first claim that if g is a twice-differentiable function on $\mathbb{R}$, then there exists $b \in \mathbb{R}$ such that $g(b)g''(b) \geq 0$. To prove this, suppose that $g(x)g''(x) < 0$ for all $x \in \mathbb{R}$. Now $g(x) \neq 0$ for all $x \in \mathbb{R}$. Since g is continuous, g has constant sign. Hence, g'' has the opposite sign. Suppose that g is positive and g'' is negative (otherwise consider $-g$ in place of g). Hence g' is decreasing, and there exists $c \in \mathbb{R}$ with $g'(c) \neq 0$. By Taylor's theorem, for each $x \in \mathbb{R}$,

$$g(x) = g(c) + (x-c)g'(c) + \frac{(x-c)^2}{2} g''(\xi),$$

where ξ is between c and x. Since g'' is negative,

$$g(x) \leq g(c) + (x-c)g'(c).$$

Depending on the sign of $g'(c)$, this implies that $g(x) < 0$ for all large enough x or for all small enough x. Either way we have a contradiction. Hence there exists $b \in \mathbb{R}$ with $g(b)g''(b) \geq 0$.

Now let f be a $(4t+3)$-times continuously differentiable function on $\mathbb{R}$. Let $F(x) = \prod_{j=0}^{4t+3} f^{(j)}(x)$. If F is always negative, then F is always nonzero, so each $f^{(j)}$ with $0 \leq j \leq 4t+3$, since it is continuous, has constant sign. From the foregoing, $f^{(j)}$ and $f^{(j+2)}$ must have the same sign for $0 \leq j \leq 4t+1$. Therefore $\prod_{j=0}^{2t+1} f^{(2j)}$ and $\prod_{j=0}^{2t+1} f^{(2j+1)}$ are both positive, so F is positive, a contradiction.

Editorial comment. The special case $t = 0$ of this problem was problem A3 on the 1998 Putnam exam.

Also solved by G. Apostolopoulos (Greece), P. P. Dályay (Hungary), J.-P. Grivaux (France), O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), J. Simons (U. K.), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), X. Wang, GCHQ Problem Solving Group (U. K.), and the proposer.

A Series Equation

11473 [2009, 941]. *Proposed by Paolo Perfetti, Mathematics Dept., University "Tor Vergata Roma," Rome, Italy.* Let α and β be real numbers such that $-1 < \alpha + \beta < 1$ and such that, for all integers $k \geq 2$,

$$-(2k)\log(2k) \neq \alpha, \qquad (2k+1)\log(2k+1) \neq \alpha,$$
$$1 + (2k+1)\log(2k+1) \neq \beta, \qquad -1-(2k+2)\log(2k+2) \neq \beta.$$

Let

$$T = \lim_{N\to\infty} \sum_{n=2}^{N} \prod_{k=2}^{n} \frac{\alpha + (-1)^k \cdot k\log(k)}{\beta + (-1)^{k+1}(1 + (k+1)\log(k+1))},$$

$$U = \lim_{N\to\infty} \sum_{n=2}^{N} ((n+1)\log(n+1)) \prod_{k=2}^{n} \frac{\alpha + (-1)^k \cdot k\log(k)}{\beta + (-1)^{k+1}(1 + (k+1)\log(k+1))}.$$

(**a**) Show that the limits defining T and U exist.
(**b**) Show that if, moreover, $|\alpha| < 1/2$ and $\beta = -\alpha$, then $T = -2U$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands.
(**a**) The series for T and U are eventually alternating in sign, so for convergence it suffices to prove that the absolute value of the term decreases eventually and converges to zero. Since $(n+1)\log(n+1)$ is an increasing function of n, it suffices to prove this for U only. The negative of the quotient of two consecutive terms is

$$\frac{(n+1)\log(n+1)}{n\log n}\cdot\frac{(-1)^n\alpha + n\log n}{1+(-1)^{n+1}\beta+(n+1)\log(n+1)}.$$

With the abbreviation $x_n = n\log n$, this expression can be written as

$$1-\frac{1-(-1)^n(\alpha+\beta)}{x_n}+\left(\frac{1}{x_{n+1}}-\frac{1}{x_n}\right)(-1)^n(\alpha+\beta)+\mathrm{O}(x_n^{-2}).$$

Since $1/x_{n+1} - 1/x_n = \mathrm{O}(n^{-1}x_n^{-1})$ and $|\alpha+\beta| < 1$, this has the form $1 - c_n$ with $1 > c_n > \frac{1}{2}(1-|\alpha+\beta|)/x_n$ eventually. Therefore $\prod_{k=1}^{n}|1-c_k|$ is eventually decreasing. Also, since $\sum x_n^{-1}$ diverges, the product goes to zero. This proves that the limit for U, and hence also for T, exists.

(**b**) The equation $T = -2U$ is incorrect. Let $p_k = (-1)^k\alpha + x_k$ and $q_k = (-1)^{k+1}\beta + 1 + x_{k+1}$. If $\alpha+\beta = 0$, then the partial sums for $T + 2U$ can be written as

$$\sum_{n=2}^{N}(-1)^{n+1}(q_n+p_{n+1})\prod_{k=2}^{n}\frac{p_k}{q_k}=\sum_{n=2}^{N}(-1)^{n+1}\left(\frac{\prod_{k=2}^{n}p_k}{\prod_{k=2}^{n-1}q_k}+\frac{\prod_{k=2}^{n+1}p_k}{\prod_{k=2}^{n}q_k}\right).$$

This is a telescoping sum that simplifies to

$$-p_2+(-1)^{N+1}p_{N+1}\prod_{k=2}^{N}\frac{p_k}{q_k}.$$

From the convergence of T and U, it follows that the second term goes to zero as N tends to infinity. Thus

$$T+2U=-\alpha-2\log 2.$$

Also solved by O. Kouba (Syria), R. Stong, and the GCHQ Problem Solving Group (U. K.).

An Inequality for Triangles

11476 [2010, 86]. *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italy.* Let a, b, and c be the side-lengths of a triangle, and let r be its inradius. Show

$$\frac{a^2bc}{(b+c)(b+c-a)}+\frac{b^2ca}{(c+a)(c+a-b)}+\frac{c^2ab}{(a+b)(a+b-c)}\ge 18r^2.$$

Solution by P. Nüesch, Lausanne, Switzerland. Write s for the semiperimeter of the triangle. The left side of the inequality is (employing geometry's cyclic summation conventions)

$$\sum\frac{a^2bc}{(b+c)(b+c-a)}=\frac{abc}{2}\sum\frac{a}{(2s-a)(s-a)}.$$

The function f defined by

$$f(x) = \frac{x}{(2s-x)(s-x)}$$

is convex for $0 < x < s$. Setting $x_1 = a$, $x_2 = b$, $x_3 = c$ yields

$$\sum \frac{a}{(2s-a)(s-a)} = \sum f(x_i) \geq 3f\left(\frac{\sum x_i}{3}\right) = 3f\left(\frac{2s}{3}\right) = \frac{9}{2s}.$$

Together with $abc = 4Rrs$ and Euler's inequality $R \geq 2r$, we obtain

$$\frac{abc}{2}\sum \frac{a}{(2s-a)(s-a)} \geq \frac{abc}{2}\,\frac{9}{2s} = 9Rr \geq 18r^2.$$

Also solved by A. Alt, G. Apostolopoulos (Greece), R. Bagby, D. Beckwith, E. Bráune (Austria), R. Chapman (U. K.), P. P. Dályay (Hungary), J. Fabrykowski & T. Smotzer, H. Y. Far, O. Faynshteyn (Germany), V. V. Garcia (Spain), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, Á. Plaza & S. Falcón (Spain), C. Pohoata (Romania), C. R. Pranesachar (India), R. Stong, E. Suppa (Italy), M. Tetiva (Romania), M. Vowe (Switzerland), L. Wimmer (Germany), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

The Winding Density of a Non-Closing Poncelet Trajectory

11479 [2010, 87]. *Proposed by Vitaly Stakhovsky, National Center for Biotechnological Information, Bethesda, MD.* Two circles are given. The larger circle C has center O and radius R. The smaller circle c is contained in the interior of C and has center o and radius r. Given an initial point P on C, we construct a sequence $\langle P_k\rangle$ (the *Poncelet trajectory* for C and c starting at P) of points on C: Put $P_0 = P$, and for $j \geq 1$, let P_j be the point on C to the right of o as seen from P_{j-1} on a line through P_{j-1} and tangent to c. For $j \geq 1$, let ω_j be the radian measure of the angle counterclockwise along C from P_{j-1} to P_j. Let

$$\Omega(C, c, P) = \lim_{k\to\infty} \frac{1}{2\pi k}\sum_{j=1}^{k} \omega_j.$$

(**a**) Show that $\Omega(C, c, P)$ exists for all allowed choices of C, c, and P, and that it is independent of P.
(**b**) Find a formula for $\Omega(C, c, P)$ in terms of r, R, and the distance d from O to o.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We will show

$$\Omega(C, c, P) = \frac{F\left(\frac{1}{2}\arccos\frac{r-d}{R}\,\Big|\, m\right)}{K(m)}, \text{ where } m = \frac{4dR}{(R+d)^2 - r^2},$$

which is independent of P. We have used the incomplete elliptic integral of the first kind, defined by

$$F(\theta|m) = \int_0^{\theta} \frac{dt}{\sqrt{1 - m\sin^2 t}} = \int_0^{\sin\theta} \frac{dy}{\sqrt{1-y^2}\,\sqrt{1-my^2}},$$

and the corresponding complete integral $K(m) = F(\pi/2|m)$.

Use coordinates with c centered at the origin and C centered on the nonnegative x-axis. Parameterize c as $T(\theta) = (r\cos\theta, r\sin\theta)$ and C as $P(\phi) = (d +$

$R\cos\phi, R\sin\phi)$. Then $\|T(\theta)\|^2 = \|T'(\theta)\|^2 = r^2$ and $\langle T'(\theta), T(\theta)\rangle = 0$. The tangent line to c at $T(\theta)$ is given by $\langle X, T(\theta)\rangle = r^2$ and a point X on the tangent can be written as

$$X = T(\theta) \pm \frac{\sqrt{\|X\|^2 - r^2}}{r} T'(\theta),$$

using the $+$ sign if X is counterclockwise from $T(\theta)$ and the $-$ sign if X is clockwise from $T(\theta)$ as viewed from the origin.

For any two points $P(\phi_1)$ and $P(\phi_2)$ on C we have

$$P(\phi_1) - P(\phi_2) = 2\sin\left(\frac{\phi_1 - \phi_2}{2}\right)\left(-R\sin\left(\frac{\phi_1+\phi_2}{2}\right), R\cos\left(\frac{\phi_1+\phi_2}{2}\right)\right),$$

$$P'(\phi_1) + P'(\phi_2) = 2\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\left(-R\sin\left(\frac{\phi_1+\phi_2}{2}\right), R\cos\left(\frac{\phi_1+\phi_2}{2}\right)\right).$$

Hence these two vectors are parallel.

For a point $T(\theta)$ on the circle c, write $P(\phi_-)$ and $P(\phi_+)$ for the two points where the tangent to c at $T(\theta)$ meet C with ϕ_+ counterclockwise from $T(\theta)$ and $\phi_- < \phi_+ < \phi_- + 2\pi$. Then $\langle P(\phi_\pm), T(\theta)\rangle = r^2$ so $\langle P(\phi_+) - P(\phi_-), T(\theta)\rangle = 0$ and hence $\langle P'(\phi_+) + P'(\phi_-), T(\theta)\rangle = 0$. Now suppose we traverse the circle c so that

$$\frac{d\theta}{dt} = \langle P'(\phi_-), T(\theta)\rangle = -\langle P'(\phi_+), T(\theta)\rangle.$$

This makes $d\theta/dt > 0$, so we traverse c in counterclockwise order. Then from

$$0 = \frac{d}{dt}\langle P(\phi_\pm), T(\theta)\rangle = \langle P'(\phi_\pm), T(\theta)\rangle \frac{d\phi_\pm}{dt} + \langle P(\phi_\pm), T'(\theta)\rangle \frac{d\theta}{dt}$$

we see

$$\begin{aligned}\frac{d\phi_\pm}{dt} &= \pm\langle P(\phi_\pm), T'(\theta)\rangle \\ &= r\sqrt{\|P(\phi_\pm)\|^2 - r^2} = r\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi_\pm}.\end{aligned}$$

Thus the elliptic integral I given by

$$I = \int_{\phi_-}^{\phi_+} \frac{d\phi}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi}}$$

satisfies

$$\begin{aligned}\frac{dI}{dt} &= \frac{1}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi_+}}\frac{d\phi_+}{dt} \\ &\quad - \frac{1}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi_-}}\frac{d\phi_-}{dt} \\ &= r - r = 0\end{aligned}$$

and is a constant. One possible chord is the vertical one through the point $(r, 0)$ with $\theta = 0$, $\phi_\pm = \pm\arccos((r-d)/R)$, so we obtain

$$\begin{aligned}I &= 2\int_0^{\arccos((r-d)/R)} \frac{d\phi}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi}} \\ &= \frac{4}{\sqrt{(R+d)^2 - r^2}}\; F\left(\frac{1}{2}\arccos\frac{r-d}{R} \,\middle|\, \frac{4dR}{(R+d)^2 - r^2}\right).\end{aligned}$$

Let

$$J = \int_0^{2\pi} \frac{d\phi}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi}}$$
$$= \frac{4}{\sqrt{(R+d)^2 - r^2}}\, K\left(\frac{4dR}{(R+d)^2 - r^2}\right).$$

Now suppose $P_0 = (d + R\cos\phi_0, R\sin\phi_0)$ and let $\phi_k = \phi_0 + \sum_{j=1}^{k} \omega_j$. We have

$$\int_{\phi_0}^{\phi_k} \frac{d\phi}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi}} = kI.$$

This integral is over an interval of at least $\lfloor(\phi_k - \phi_0)/(2\pi)\rfloor$ complete periods and fewer than $\lceil(\phi_k - \phi_0)/(2\pi)\rceil$ complete periods. Hence

$$\left\lfloor \frac{\sum_{j=1}^{k} \omega_j}{2\pi} \right\rfloor J \le kI \le \left\lceil \frac{\sum_{j=1}^{k} \omega_j}{2\pi} \right\rceil J.$$

Thus

$$\frac{I}{J} - \frac{1}{k} \le \frac{1}{k}\left(\left\lceil \frac{\sum_{j=1}^{k} \omega_j}{2\pi} \right\rceil - 1\right) \le \frac{\sum_{j=1}^{k} \omega_j}{2\pi k} \le \frac{1}{k}\left(\left\lfloor \frac{\sum_{j=1}^{k} \omega_j}{2\pi} \right\rfloor + 1\right) \le \frac{I}{J} + \frac{1}{k}$$

and

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{k} \omega_j}{2\pi k} = \frac{I}{J},$$

which is the quotient of elliptic integrals claimed.

Editorial comment.

In the classical case, when the trajectory closes—returns to its starting point after finitely many steps—this "winding density" is rational: the number of times the closed trajectory goes around the circle divided by the number of intervals in the trajectory. The use of elliptic integrals to compute it is known, and in many special cases it can be computed without elliptic integrals: see `http://mathworld.wolfram.com/PonceletsPorism.html`.

Also solved by J. A. Grzesik, and the proposer.