Junior problems

J67. Prove that among seven arbitrary perfect squares there are two whose difference is divisible by 20.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

J68. Let ABC be a triangle with circumradius R. Prove that if the length of one of the medians is equal to R, then the triangle is not acute. Characterize all triangles for which the lengths of two medians are equal to R.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

J69. Consider a convex polygon $A_1A_2\ldots A_n$ and a point P in its interior. Find the least number of triangles $A_iA_jA_k$ that contain P on their sides or in their interiors.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

J70. Let l_a, l_b, l_c be the lengths of the angle bisectors of a triangle. Prove the following identity

$$\frac{\sin\frac{\alpha-\beta}{2}}{l_c}+\frac{\sin\frac{\beta-\gamma}{2}}{l_a}+\frac{\sin\frac{\gamma-\alpha}{2}}{l_b}=0,$$

where α, β, γ are the angles of the triangle.

Proposed by Oleh Faynshteyn, Leipzig, Germany

J71. In the Cartesian plane call a line "good" if it contains infinitely many lattice points. Two lines intersect at a lattice point at an angle of $45°$ degrees. Prove that if one of the lines is good, then so is the other.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

J72. Let a, b, c be real numbers such that $|a|^3 \le bc$. Prove that $b^2+c^2 \ge \frac{1}{3}$ whenever $a^6+b^6+c^6 \ge \frac{1}{27}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior problems

S67. Let ABC be a triangle. Prove that

$$\cos^3 A + \cos^3 B + \cos^3 C + 5\cos A\cos B\cos C \leq 1.$$

Proposed by Daniel Campos Salas, Costa Rica

S68. Let ABC be an isosceles triangle with $AB = AC$. Let X ad Y be points on sides BC and CA such that $XY \parallel AB$. Denote by D the circumcenter of triangle CXY and by E be the midpoint of BY. Prove that $\angle AED = 90°$.

Proposed by Francisco Javier Garcia Capitan, Spain

S69. Circles ω_1 and ω_2 intersect at X and Y. Let AB be a common tangent with $A \in \omega_1$, $B \in \omega_2$. Point Y lies inside triangle ABX. Let C and D be the intersections of an arbitrary line, parallel to AB, with ω_1 and ω_2, such that $C \in \omega_1$, $D \in \omega_2$, C is not inside ω_2, and D is not inside ω_1. Denote by Z the intersection of lines AC and BD. Prove that XZ is the bisector of angle CXD.

Proposed by Son Hong Ta, Ha Noi University, Vietnam

S70. Find the least odd positive integer n such that for each prime p, $\frac{n^2-1}{4} + np^4 + p^8$ is divisible by at least four primes.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S71. Let ABC be a triangle and let P be a point inside the triangle. Denote by $\alpha = \frac{\angle BPC}{2}, \beta = \frac{\angle CPA}{2}, \gamma = \frac{\angle APB}{2}$. Prove that if I is the incenter of ABC, then

$$\frac{\sin\alpha\sin\beta\sin\gamma}{\sin A\sin B\sin C} \geq \frac{R}{2(r+PI)},$$

where R and r are the circumcenter and incenter, respectively.

Proposed by Khoa Lu Nguyen, Massachusetts Institute of Technology, USA

S72. Let ABC be a triangle and let $\omega(I)$ and $C(O)$ be its incircle and circumcircle, respectively. Let D, E, and F be the intersections with $C(O)$ of the lines through I perpendicular to sides BC, CA and AB, respectively. Two triangles XYZ and $X'Y'Z'$, with the same circumcircle, are called *parallelopolar* if and only if the Simson line of X with respect to triangle $X'Y'Z'$ is parallel to YZ and two analogous relations hold. Prove that triangles ABC and DEF are parallelopolar.

Proposed by Cosmin Pohoata, Bucharest, Romania

Undergraduate problems

U67. Let $(a_n)_{n\geq 0}$ be a decreasing sequence of positive real numbers. Prove that if the series $\sum_{k=1}^{\infty} a_k$ diverges, then so does the series $\sum_{k=1}^{\infty}\left(\frac{a_0}{a_1}+\cdots+\frac{a_{k-1}}{a_k}\right)^{-1}$.

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

U68. In the plane consider two lines d_1 and d_2 and let $B, C \in d_1$ and $A \in d_2$. Denote by M the midpoint of BC and by A' the orthogonal projection of A onto d_1. Let P be a point on d_2 such that $T = PM \cap AA'$ lies in the halfplane bounded by d_1 and containing A. Prove that there is a point Q on segment AP such that the angle bisector of Q passes through T.

Proposed by Nicolae Nica and Cristina Nica, Romania

U69. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\arctan\frac{k}{n}\right)\sin\frac{1}{n+k}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

U70. For all integers $k, n \geq 2$ prove that

$$\sqrt[n]{1+\frac{n}{k}} \leq \frac{1}{n}\log\left(1+\frac{n-1}{k-1}\right)+1.$$

Proposed by Oleg Golberg, Massachusetts Institute of Technology, USA

U71. A polynomial $p \in \mathbb{R}[X]$ is called a "mirror" if $|p(x)| = |p(-x)|$. Let $f \in \mathbb{R}[X]$ and consider polynomials $p, q \in \mathbb{R}[X]$ such that $p(x) - p'(x) = f(x)$, and $q(x) + q'(x) = f(x)$. Prove that $p + q$ is a mirror polynomial if and only if f is a mirror polynomial.

Proposed by Iurie Boreico, Harvard University, USA

U72. Let n be an even integer. Evaluate

$$\lim_{x\to -1}\left[\frac{n(x^n+1)}{(x^2-1)(x^n-1)}-\frac{1}{(x+1)^2}\right].$$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

Olympiad problems

O67. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that for $a > 0$,

$$a + a_1^2 + a_2^2 + \ldots + a_n^2 \geq m(a_1 + a_2 + \ldots + a_n),$$

where $m = 2\sqrt{\frac{a}{n}}$, if n is even, and $m = 2\sqrt{\frac{an}{n^2-1}}$, if n is odd.

Proposed by Pham Kim Hung, Stanford University, USA

O68. Let $ABCD$ be a quadrilateral and let P be a point in its interior. Denote by K, L, M, N the orthogonal projections of P onto lines AB, BC, CD, DA, and by H_a, H_b, H_c, H_d the orthocenters of triangles AKN, BKL, CLM, DMN, respectively. Prove that H_a, H_b, H_c, H_d are the vertices of a parallelogram.

Proposed by Mihai Miculita, Oradea, Romania

O69. Find all integers a, b, c for which there is a positive integer n such that

$$\left(\frac{a + bi\sqrt{3}}{2}\right)^n = c + i\sqrt{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

O70. In triangle ABC let M_a, M_b, M_c be the midpoints of BC, CA, AB, respectively. The incircle (I) of triangle ABC touches the sides BC, AC, AB at points A', B', C'. The line r_1 is the reflection of line BC in AI, and line r_2 is the perpendicular from A' to IM_a. Denote by X_a the intersection of r_1 and r_2, and define X_b and X_c analogously. Prove that X_a, X_b, X_c lie on a line that is tangent to the incircle of triangle ABC.

Proposed by Jan Vonk, Ghent University, Belgium

O71. Let n be a positive integer. Prove that $\displaystyle\sum_{k=1}^{n-1} \frac{1}{\cos^2 \frac{k\pi}{2n}} = \frac{2}{3}(n^2 - 1).$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

O72. For $n \geq 2$, let S_n be the set of divisors of all polynomials of degree n with coefficients in $\{-1, 0, 1\}$. Let $C(n)$ be the greatest coefficient of a polynomial with integer coefficients that belongs to S_n. Prove that there is a positive integer k such that for all $n > k$,

$$n^{2007} < C(n) < 2^n.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Gabriel Dospinescu, Ecole Normale Superieure, France