Junior problems

J133. A sequence $(a_n)_{n\geq 2}$ of real numbers greater than 1 satisfies the relation

$$a_n = \sqrt{1 + \frac{(n+1)!}{2\left(a_2 - \frac{1}{a_2}\right)\cdots\left(a_{n-1} - \frac{1}{a_{n-1}}\right)}}$$

for all $n > 2$. Prove that if $a_k = k$ for some $k \geq 2$, then $a_n = n$ for all $n \geq 2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J134. How many positive integers n less than 2009 are divisible by $\lfloor \sqrt[3]{n} \rfloor$?

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

J135. Find all n for which the number of diagonals of a convex n-gon is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J136. Let a, b, c be the sides, m_a, m_b, m_c the medians, h_a, h_b, h_c the altitudes, and l_a, l_b, l_c the angle bisectors of a triangle ABC. Prove that the diameter of the circumcircle of triangle ABC is equal to

$$\frac{l_a^2}{h_a}\sqrt{\frac{m_a^2 - h_a^2}{l_a^2 - h_a^2}}.$$

Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Bari, Italy

J137. Let ABC be a triangle and let tangents to the circumcircle at A, B, C intersect BC, AC, AB at points A_1, B_1, C_1, respectively. Prove that

$$\frac{1}{AA_1} + \frac{1}{BB_1} + \frac{1}{CC_1} = 2\max\left(\frac{1}{AA_1}, \frac{1}{BB_1}, \frac{1}{CC_1}\right).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J138. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2+c^2} + \frac{b^3}{c^2+a^2} + \frac{c^3}{a^2+b^2} \geq \frac{a+b+c}{2}.$$

Proposed by Mircea Becheanu, University of Bucharest, Romania

Senior problems

S133. There are 144 lilypads in a row and are colored red, green, blue, red, green, blue and so on. Prove that the number of ways for a frog to reach the last lilypad from the first lilypad in a sequence of left-to-right jumps between lilypads of different color is a multiple of 3.

Proposed by Brian Basham, Massachusetts Institute of Technology, USA

S134. Find all triples (x, y, z) of integers satisfying the system of equations

$$\begin{cases} x + y = 5z \\ xy = 5z^2 + 1. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S135. Zeroes are written at every vertex of a regular $n-$gon. Every minute, Bob picks a vertex, adds 2 to the number written at that vertex, and subtracts 1 from the numbers written at the two adjacent vertices. Prove that, no matter how long Bob plays, he will never be able to achieve a configuration in which a 1 is written at one vertex, a -1 is written at another, and a zero is written everywhere else.

Proposed by Timothy Chu, Lynbrook Highschool, USA

S136. A weightlifter lifts a barbell which has n equal weights on both of its sides. At each step he takes out weights from one of two sides of the barbell. However, if the difference between the weights on the sides is greater than k weights, the barbell will turn and fall. What is the least number of steps required to take out all the weights?

Proposed by Iurie Boreico, Harvard University, USA

S137. Let $ABCD$ be a cyclic quadrilateral and let $\{U\} = AB \cap CD$ and $\{V\} = BC \cap AD$. The line that passes through V and is perpendicular to the angle bisector of angle $\angle AUD$ intersects UA and UD at X and Y, respectively. Prove that

$$AX \cdot DY = BX \cdot CY.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S138. Let a, b, c be positive real numbers such that $\sqrt{a} + \sqrt{b} + \sqrt{c} = 3$. Prove that

$$8(a^2 + b^2 + c^2) \geq 3(a + b)(b + c)(c + a).$$

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Undergraduate problems

U133. Let f be a continuous real valued function defined on $[0,1]$ such that $\int_0^1 f(x)dx = \int_0^1 xf(x)dx$. Prove that there is a real number $c \in (0,1)$ for which $c \cdot f(c) = 2\int_c^0 f(x)dx$.

Proposed by Duong Viet Thong, Nam Dinh University, Vietnam

U134. Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $f(x_1) + f(x_2) \geq 2f(x_1 + x_2)$ for all $x_1, x_2 \geq 0$. Prove that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(x_1 + x_2 + \cdots + x_n)$$

for all $x_1, x_2, \ldots, x_n \geq 0$.

Proposed by Mihai Piticari, Romania

U135. Suppose that $f, g : (0,\infty) \to (a,\infty)$ are continuous convex functions such that f is increasing and continuously differentiable. Prove that if

$$f'(x) \geq \frac{f(g(x)) - f(x)}{x}$$

for all $x > 0$, then $g(x) \leq 2x$ for all $x > 0$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

U136. Let P be a non-constant polynomial. Prove that there are infinitely many positive integers n such that $(P(n))^n$ is not a power of a prime.

Proposed by Cezar Lupu, University of Bucharest, Romania

U137. Suppose that k and n are positive integers with $n > 1$ and that $A_1, A_2, \ldots, A_k$, $B_1, B_2, \ldots, B_k$ are $n \times n$ matrices with real entries such that for each matrix X with real entries satisfying $X^2 = O_n$, the matrix $A_1XB_1 + A_2XB_2 + \cdots + A_kXB_k$ is nilpotent. Prove that $A_1B_1 + A_2B_2 + \cdots + A_kB_k$ is of the form aI_n for some real number a.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

U138. Let q be a Fermat prime and let $n \leq q$ be a positive integer. Let p be a prime divisor of $1 + n + \cdots + n^{q-1}$. Define a function λ on real numbers x by $\lambda(x) = x - \frac{x^2}{2} + \cdots - \frac{x^{p-1}}{p-1}$. Prove that p divides the numerator of the fraction

$$\sum_{j=0}^{\log_2 \frac{q-1}{2}} \frac{\lambda(n^{2^j})(n^{pq-p} - 1)}{(n^p - 1)(n^{2^j p} + 1)}$$

when it is written in lowest terms.

Proposed by David B. Rush, Massachusetts Institute of Technology, USA

Olympiad problems

O133. Let a, b, c and x, y, z be positive real numbers such that $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \sqrt[3]{m}$ and $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{n}$. Prove that

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \geq \frac{m}{n}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O134. Let p be a prime and let n be an integer greater than 4. Prove that if a is an integer that is not divisible by p, then the polynomial $ax^n - px^2 + px + p^2$ is irreducible in $\mathbb{Z}[X]$.

Proposed by Mircea Becheanu, University of Bucharest, Romania

O135. In a convex quadrilateral $ABCD$, $AC \cap BD = \{E\}, AB \cap CD = \{F\}$, and EF intersects the sides AD and BC at X and Y. Let M and N be the midpoints of AD and BC, respectively. Prove that quadrilateral $BCMX$ is cyclic if and only if quadrilateral $ADNY$ is cyclic.

Proposed by Andrei Ciupan, Tudor Vianu High School, Romania

O136. For a positive integer n and a prime p, denote by $v_p(n)$ the nonnegative integer for which $p^{v_p(n)}$ divides n but $p^{v_p(n)+1}$ does not. Prove that $v_5(n) = v_5(F_n)$, where F_n denotes the n^{th} Fibonacci number.

Proposed by David B. Rush, Massachusetts Institute of Technology, USA

O137. Find the locus of centers of the equilateral triangles inscribed in a given square.

Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Sofia

O138. Consider a regular hexagon with side 1. There are only two ways to tile this hexagon with rhombi with side 1. Each of these two tilings involve three rhombi of different types. Prove that no matter how we tile a regular hexagon of side n with rhombi with side 1, the number of rhombi of each type is the same.

Proposed by Ivan Borsenco, MIT and Iurie Boreico, Harvard University, USA