Problem 1233 di "Pi Mu Epsilon" Fall 2010

Evaluate

$$\sum_{k=1}^{\infty} \frac{k(2k+1)H_{k-1}^2 + k(1+k)^2 H_{k-1} + (1+k)^2}{k^3(k+1)^2}$$

$H_k = 1 + 1/2 + \ldots + 1/k$, $H_0 = 0$.

Answer: 3.

Proof

$$\frac{k(2k+1)H_{k-1}^2 + k(1+k)^2 H_{k-1} + (1+k)^2}{k^3(k+1)^2} = \frac{2k+1}{k^2(k+1)^2} H_{k-1}^2 + \frac{1}{k^2}(H_{k-1} + \frac{1}{k}) =$$
$$= \frac{2k+1}{k^2(k+1)^2} H_{k-1}^2 + \frac{H_k}{k^2} \doteq S$$

Moreover

$$\frac{2k+1}{k^2(k+1)^2} H_{k-1}^2 = H_{k-1}^2 \Big(\frac{1}{k^2} - \frac{1}{(1+k)^2} \Big) =$$
$$= \Big(\frac{H_k^2}{k^2} - 2\frac{H_{k-1}}{k^3} - \frac{1}{k^4} \Big) + \Big(-\frac{H_k^2}{(k+1)^2} + 2\frac{H_{k-1}}{k(k+1)^2} + \frac{1}{k^2(1+k^2)} \Big)$$

We make use of the following equalities 1) and 2) (see for example the article D.Borwein and J.M.Borwein *On an intriguing Integral and some Series Related to* $\zeta(4)$ Proceedings of the Amer. Math. Soc., Vol 123, No.4 (Apr.,1995) pp.1191–1198)

$$1) \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} = \frac{17}{4}\zeta(4), \quad 2) \sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)^2} = \frac{11}{4}\zeta(4),$$
$$3) \sum_{k=1}^{\infty} \frac{H_k}{k^3} = \frac{5}{4}\zeta(4), \quad 4) \sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3)$$

3) can be obtained by 1) and 2) because $H_{k-1} = H_k - \frac{1}{k}$ and then $H_{k-1}^2 = H_k^2 - 2\frac{H_k}{k} + \frac{1}{k^2}$ hence

$$2\sum_{k=1}^{\infty} \frac{H_k}{k^3} = \sum_{k=1}^{\infty} \left(\frac{1}{k^4} + \frac{H_k^2}{k^2} - \frac{H_{k-1}^2}{k^2} \right) = \zeta(4) \left(1 + \frac{17}{4} - \frac{11}{4} \right) = \frac{5}{2}\zeta(4)$$

4) is an old result going back to Euler yielding

$$\sum_{k=1}^{\infty} \frac{H_{k+1}}{(k+1)^2} = \sum_{k=1}^{\infty} \frac{H_k}{k^2} - 1 = 2\zeta(3) - 1$$

Moreover

$$\sum_{k=1}^{\infty} \frac{H_{k-1}}{k^3} = \sum_{k=1}^{\infty} \frac{H_k}{k^3} - \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\zeta(4)}{4}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2(1+k^2)}=\sum_{k=1}^{\infty}\Big(\frac{1}{k^2}+\frac{1}{(1+k)^2}+\frac{2}{k+1}-\frac{2}{k}\Big)=\zeta(2)+(\zeta(2)-1)-2=2\zeta(2)-3.$$

$$\frac{H_{k-1}}{k(1+k)^2}=\frac{H_{k-1}}{k}-\frac{H_{k-1}}{k+1}-\frac{H_{k-1}}{(1+k)^2}=$$
$$=\Big[\frac{H_k}{k}-\frac{1}{k^2}\Big]-\Big[\frac{H_{k+1}}{k+1}-\frac{1}{(k+1)^2}-\frac{1}{k(k+1)}\Big]-\Big[\frac{H_{k+1}}{(1+k)^2}-\frac{1}{(1+k)^3}-\frac{1}{k(1+k)^2}\Big]=$$
$$=\Big[\frac{H_k}{k}-\frac{H_{k+1}}{k+1}\Big]+\Big[\frac{1}{(k+1)^2}-\frac{1}{k^2}\Big]+\frac{1}{k(k+1)}-\frac{H_{k+1}}{(1+k)^2}+\frac{1}{(1+k)^3}+\frac{1}{k(1+k)^2}$$

Telescoping

$$\sum_{k=1}^{\infty}\Big(\frac{H_k}{k}-\frac{H_{k+1}}{k+1}+\frac{1}{(k+1)^2}-\frac{1}{k^2}\Big)=0$$

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=\sum_{k=1}^{\infty}(\frac{1}{k}-\frac{1}{(k+1)})=1,$$

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)^2}=\sum_{k=1}^{\infty}(\frac{1}{k}-\frac{1}{k+1}-\frac{1}{(k+1)^2})=1-(\zeta(2)-1)=2-\zeta(2)$$

hence

$$\sum_{k=1}^{\infty}\frac{H_{k-1}}{k(1+k)^2}=1-(2\zeta(3)-1)+\zeta(3)-1+2-\zeta(2)=3-\zeta(3)-\zeta(2)$$

Summing all the contributions we get

$$S=\zeta(4)(\frac{17}{4}-2\frac{1}{4}-1-\frac{11}{4})+2(3-\zeta(3)-\zeta(2))+2\zeta(2)-3+2\zeta(3)=3$$