Paolo Perfetti, Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", facoltà di Ingegneria

# Problems I cannot solve (at the moment)

**If anyone solves some of them, I would like to know the solution** perfetti@mat.uniroma2.it

**1)** Write a polynomial $P(x,y)$, $(x,y)\in \mathbf{R}^2$, with three or more critical points all maxima (no saddle) [It is known an example with two critical points, A. Durfee, N.Kronenfeld, H.Munson, J.Roy, I.Westby American Mathematical Monthly 100, No.3 (Mar., 1993), 255-271]

**2)** Is the everywhere divergent trigonometric series (Steinhaus) $\displaystyle\sum_{k=2}^{\infty}\frac{\sin(k(x-\ln\ln k))}{\ln k}$ a Fourier series?

H.Steinhaus: *A divergent trigonometrical series,* Journal of London Mathematical Society, **4** (1929) pp.86–88, Sioban O'Shea: *On a Divergent Trigonometrical Series Given by Steinhaus*, Proocedings of the American Mathematical Society, Volume 10, Issue 1 (Feb., 1959) pp.60–70. A.Zygmund: *Trigonometric Series*, Third Edition, Volume–I, (Cambridge Mathematical Library), p.338. Some years ago Elias Stein e–mailed me that if Zygmund has written it is not Fourier, he certainly knew the proof.

**3)** Does the series $\displaystyle\sum_{k=3}^{\infty}\frac{(-1)^k}{\ln k+\sin k}$ converge?

**4)** $\{d_k\}$ is a monotone decreasing sequence of positive numbers such that $\sum d_k=+\infty$. Prove or disprove: a monotone decreasing sequence $\{d'_k\}$ of positive terms can be found such that $\sum d'_k=+\infty$ but $\sum\min\{d_k,d'_k\}<+\infty$.

**5)** Let $\{a_k\}=\{1/(k\ln(k+1))\}$, $k\geq 1$. Let $\{b_k\}$ be another sequence defined by $b_1=1$, $b_2=-a_2$, $b_3=-a_3$, $b_4=a_4$, $b_5=a_5$, $b_6=a_6$, $b_7=a_7$, $b_8=-a_8$, $\dots$ $b_{15}=-a_{15}$ and so on. The series $\sum b_k$ converges but diverges absolutely. Moreover $\displaystyle\sum b_k z^k$ $(z\in\mathbf{C})$ converges on $|z|\leq 1$. Is the convergence uniform?

**6)** Let $\{a_k\}$ be a sequence of nonnegative real numbers such that $\displaystyle\lim_{n\to\infty}\sum^{n}a_k\doteq\lim_{n\to\infty}A_n=\infty$. Let $\{g_k\}$ be a sequence such that $g_k\to+\infty$ and $\displaystyle\sum\frac{a_k}{A_k g_k}=+\infty$.

Give a monotonic sequence $\{f_n\}$ $f_n\leq f_{n+1}$ such that $\displaystyle\sum\frac{a_k}{f_k}=+\infty$, but $\displaystyle\sum\min\{\frac{a_k}{A_k g_k},\frac{a_k}{f_k}\}<+\infty$, [*Paul Erdos; N.J.Fine; J.B.Kelly – A.M.M.* Vol.58, No.6 (Jun. – Jul., 1951), 425–426]

**7)** (**A question raised by Littlewood**) Let $\{a_k\}$ be a sequence such that $0<a_k\leq a_{k+1}\leq k+1$. It is known (AMM Vol.72, No.6, (Jun. – Jul., 1965), pp.675–677) that the series $\sum(a_k/A_k)^\alpha$ converges if $\alpha>1$ and $A_n=a_1+\ldots+a_n$. Littlewood raised the question about the convergence of the series if $a_k\leq k\omega_k$ with $\omega_k\to\infty$ sufficiently slow.

**8)** For $x\in(0,\pi/2]$ prove $\displaystyle\frac{\sin^4 x}{x^4}+\frac{\sin x}{13}\geq\cos x$

  