Solution

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1 Introduction

We use the following bounds: for all $x \in (0, \pi/2]$,

$$\sin x \geq x - x^3/6,$$

$$\cos x \leq 1 - x^2/2 + x^4/24,$$

and

$$\sin^4 x \geq x^4 - 2x^6/3 + x^8/5 - 34x^{10}/945.1$$

(The proof of (1) is given at the end. The RHS of (1) is the Taylor approximation of $\sin^4 x$ around $x = 0$.)

It suffices to prove that, for all $x \in (0, \pi/2]$,

$$1 - 2x^2/3 + x^4/5 - 34x^6/945 + \frac{x - x^3/6}{13} \geq 1 - x^2/2 + x^4/24,$$

or

$$\frac{1}{13}x - \frac{1}{6}x^2 - \frac{1}{78}x^3 + \frac{19}{120}x^4 - \frac{34}{945}x^6 \geq 02$$

which is true. The proof of (2) is given at the end.

We are done.

Proof of (1).

Using $\sin x \geq x - x^3/6 + x^5/120 - x^7/5040 \geq 0$ for all $x \in (0, \pi/2]$, by Bernoulli inequality, we have

$$\sin^4 x \geq (x - x^3/6 + x^5/120 - x^7/5040)^4$$

$$= (x - x^3/6)^4 \left(1 + \frac{x^5/120 - x^7/5040}{x - x^3/6}\right)^4$$

$$\geq (x - x^3/6)^4 \left(1 + \frac{x^5/120 - x^7/5040}{x - x^3/6} \cdot 4\right)$$

$$= x^4 - 2x^6/3 + x^8/5 - 34x^{10}/945 + \frac{1}{272160}(x^4 - 60x^2 + 1074)x^2$$

$$\geq x^4 - 2x^6/3 + x^8/5 - 34x^{10}/945.$$

We are done.

Proof of (2).

It suffices to prove that

$$\frac{1}{13} - \frac{1}{6}x - \frac{1}{78}x^2 + \frac{19}{120}x^3 - \frac{34}{945}x^5 \geq 0.$$

Using $x^3 \leq 3x^2 - 8x/3 + 20/27$ (equivalent to $(x-2/3)^2(5/3-x) \geq 0$), it suffices to prove that

$$\frac{1}{13} - \frac{1}{6}x - \frac{1}{78}x^2 + \frac{19}{120}x^3 - \frac{34}{945}x^2 \cdot (3x^2 - 8x/3 + 20/27) \geq 0.$$

(**Note**: Now, the polynomial is of degree four.)

Do the same things twice again, it suffices to prove that

$$\frac{105509}{2653560}x^2 - \frac{347}{5670}x + \frac{10117}{398034} \geq 0$$

which is true. We are done.