József Wildt International Mathematical Competition

The Edition XXIVth , 2014 [1]

The solution of the problems W.1 - W.41 must be mailed before 30. September 2014, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania, E-mail: benczemihaly@yahoo.com

W1. Let a, $b \in \mathbb{N}$, with $b \geq 2$. If $a \not\equiv 0 \pmod{b}$, then

$$\left\{\frac{a}{b}\right\} = \left\{\frac{a-1}{b}\right\} + \frac{1}{b}\,.$$

If $a \equiv 0 \pmod{b}$, then

$$\left\{\frac{a-2}{b}\right\} = 1 - \frac{2}{b}\,.$$

Michael Th. Rassias

W2. Let V_1, V_2 and V_3 be three subspaces of a vectorial space V of dimension n. Prove that

$$\frac{1}{n}\Big(\dim(V_1) + \dim(V_2) + \dim(V_3) - \dim(V_1 \cap V_2 \cap V_3)\Big) \leq 2$$

José Luis Díaz-Barrero

W3. Let A be a 3×3 real orthogonal matrix with $\det(A) = 1$. Compute

$$(trA - 1)^2 + \sum_{i<j} (a_{ij} - a_{ji})^2$$

José Luis Díaz-Barrero

W4. A sequence of integers $\{a_n\}_{n\geq 1}$ is given by the conditions $a_1 = 1, a_2 = 12, a_3 = 20$, and $a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n$ for every $n \geq 1$. Prove that for every positive integer n, the number $1 + 4a_n a_{n+1}$ is a perfect square.

José Luis Díaz-Barrero

W5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \ln(x + \sqrt{1+x^2}) + (1+x^2)^{-1/2} - x\,(1+x^2)^{-3/2}$$

and suppose that for the reals $a < b$ is $\ln\left(\dfrac{f(b)}{f(a)}\right) = b - a$. Show that there exists $c \in (a, b)$ for which holds

$$2c^2 = 1 + (1+c^2)^{5/2} \ln(c + \sqrt{1+c^2})$$

José Luis Díaz-Barrero

W6. Let D_1 be set of strictly decreasing sequences of positive real numbers with first term equal to 1.For any $\mathbf{x}_{\mathbb{N}} := (x_1, x_2, ..., x_n, ...) \in D_1$ prove that

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} \geq \frac{4}{9}$$

and find the sequence for which equality occurs.

Arkady Alt

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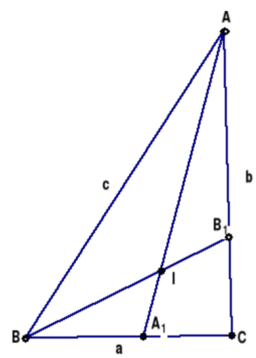
W7. Let $\triangle ABC$ be a right triangle with right angle in C and let be intersection point of bisectors AA_1, BB_1 of acute angles $\angle A$ and $\angle B$, respectively.
Find the right triangle with greatest value of ratio of the "bisectoria" quadrilateral A_1CB_1I area to the triangle $\triangle ABC$ area.



Arkady Alt

W8. Let

$$\Delta(x, y, z) = 2xy + 2yz + 2zx - x^2 - y^2 - z^2.$$

Find all triangles with sidelengths a, b, c such that $\Delta(a^n, b^n, c^n) > 0$ for any $n \in \mathbb{N}$.

Arkady Alt

W9. Let R, r and s be, respectively,circumradius, inradius and semiperimeter of a triangle.
a) Prove inequality $R^2 - 4r^2 \geq \frac{1}{5} \cdot (s^2 - 27r^2)$;
b) Find the maximum value for constant K such that inequality $R^2 - 4r^2 \geq K(s^2 - 27r^2)$ holds for any triangle;
c) Find the $\lim\limits_{R \to 2r} \frac{R^2 - 4r^2}{s^2 - 27r^2}$.

Arkady Alt

W10. Let a, b and c be positive numbers such that $a^2 + b^2 + c^2 + 2abc = 1$. Prove that

$$\sum_{\text{cyc}} \sqrt{a\left(\frac{1}{b} - b\right)\left(\frac{1}{c} - c\right)} \geq \frac{3\sqrt{3}}{2} \sqrt{\sum_{\text{cyc}} \frac{c(ab + c)}{2abc + a^2 + c^2}}$$

Paolo Perfetti

W11. Let a, b and c be positive numbers such that $a^2 + b^2 + c^2 + 2abc = 1$. Prove that

$$\sum_{\text{cyc}} \sqrt{a\left(\frac{1}{b} - b\right)\left(\frac{1}{c} - c\right)} \geq \left(\frac{3}{2}\right)^{3/2} \sqrt{\sum_{\text{cyc}} \frac{c(ab + c)(2abc + a^2 + b^2)}{a(bc + a)(2abc + c^2 + b^2)}}$$

Paolo Perfetti

W12. Evaluate

$$\int_0^{\pi/2} (\ln(1 + \tan^4 \vartheta))^2 \frac{2\cos^2 \vartheta}{2 - (\sin(2\vartheta))^2} d\vartheta$$

Answer: $-\frac{4\pi}{\sqrt{2}}C + \frac{13\pi^3}{24\sqrt{2}} + \frac{9}{2}\frac{\pi \ln^2 2}{\sqrt{2}} - \frac{3}{2}\frac{\pi^2 \ln 2}{\sqrt{2}}$

$C=\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}=\int_0^1\frac{\arctan x}{x}dx\sim 0,915916$ is the Catalan constant.

Paolo Perfetti

W13. Show that $x^x-1\leq xe^{x-1}(x-1)$ for $0\leq x\leq 1$.

Paolo Perfetti

W14. Calculate

$$\int_0^1\int_0^1\ln(1-x)\ln(1-xy)dxdy.$$

Ovidiu Furdui

W15. Calculate

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}-\ln\left(k+\frac{1}{2}\right)-\gamma\right).$$

Ovidiu Furdui

W16. Calculate

$$\int_0^1 x\ln\left(\sqrt{1+x}-\sqrt{1-x}\right)\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)dx.$$

Ovidiu Furdui

W17. Let $a\in R_{+,\ b,c\in(1,\infty)}$ and $f,g:R\to R$ be continuous and odd functions. Prove that:

$$\int\limits_{-a}^{a}f(x)\ln\left(b^{g(x)}+c^{g(x)}\right)dx=\ln(bc)\int\limits_{0}^{a}f(x)\,g(x)\,dx.$$

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W18. Prove that if $m,n\in(0,\infty)$, then in any triangle ABC with usual notations holds:

$$\frac{ma^2+nb^2}{a+b-c}+\frac{mb^2+nc^2}{b+c-a}+\frac{mc^2+na^2}{c+a-b}\geq 2\,(m+n)\,s$$

D.M. Bătineţu-Giurgiu, Neculai Stanciu and Titu Zvonaru

W19. If $n\in N$, $n\geq 3$, $a,b,c,d,x_k\in R_k^*$, $k=\overline{1,n}$, $x_{n+1}=x_1$ such that

$$\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\prod_{k=1}^{n}x_k\leq d,$$

then prove that:

$$\sum_{k=1}^{n}\left(ax_k^{n-1}+bx_k^{n-1}+c\right)\frac{x_k^3+x_{k+1}^3}{x_k^2+x_kx_{k+1}+x_{k+1}^2}\geq\frac{2n}{3d}\left((a+b)\,d+cn\right)\prod_{k=1}^{n}x_k$$

D.M. Bătineţu-Giurgiu, Neculai Stanciu and Titu Zvonaru

W20. Let a, b, and c denote the side lengths of a triangle. Show that

$$\sum_{cyc}\frac{40bc}{5a^2-5(b+c)^2+48bc}\leq\frac{29}{11}+\sum_{cyc}\frac{bc}{(b+c)^2-a^2}.$$

Pál Péter Dályay

W21. Let m and n be positive integers, and let A_1, A_2, $\ldots$, A_m be open subsets of R, each of them with n connected components such that for any $1 \leq i < j \leq m$ we have $A_i \cap A_j \neq$. Show that if $m = 2n+1$, then there exist three different positive integers i, j, and k not greater than m such that $A_i \cap A_j \cap A_k \neq$.

Pál Péter Dályay

W22. Let R, r, and s be the circumradius, the inradius, and the semiperimeter of a triangle, respectively. Show that

$$(4R+r)^3 \geq s^2(16R-5r).$$

When holds the equality?

Pál Péter Dályay

W23. A, B matrices in $M_n(\mathbb{C})$, $C = AB - BA$ suppose $AC = CA$, $BC = BC$

$$\forall t \in \mathbb{R}, m(t) = \exp\left(-t\left(A+B\right)\right)\exp\left(tA\right)\exp\left(tB\right)$$

Express $m(t)$ only with C

Moubinool Omarjee

W24. Find all $(x, y, z) \in \mathbb{Q}^3$, such that

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 0$$

Moubinool Omarjee

W25. $y \in \mathbb{Q}$, $y^2 \in \mathbb{N}$ find the radius of convergence of the power series

$$\sum_{n \geq 1} \frac{x^n}{\left|\sin\left(n\pi y\right)\right|}$$

Moubinool Omarjee

W26. Let $ABCD$ be a cyclic quadrilateral. We note that $AC = e$ and $BD = f$. Denoted by r_a, r_b, r_c respectively r_d the radii of the incircles of the triangles BCD, CDA, DAB respectively ABC. Prove the following equality:

$$e\left(r_a^2 + r_c^2\right) = f\left(r_b^2 + r_d^2\right)$$

Nicuşor Minculete and Cătălin Barbu

W27. In any convex quadrilateral $ABCD$ with lengths of sides given as $AB = a$, $BC = b$, $CD = c$ respectiv $DA = d$ and S the area. Prove that

$$(3a+b+c+d)^n + (a+3b+c+d)^n + (a+b+3c+d)^n +$$

$$+(a+b+c+3d)^n \geq 2^{\frac{n+8}{2}} \cdot 3^n \cdot \sqrt{S}$$

for every $n \in N^{*}$.

Nicuşor Minculete

W28. For $x, y, z \in R^*$, we note

$$E(x,y,z) = (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

a). If $x, y, z \in R$ so that $x \cdot y \cdot z > 0$ and $\min(x;y;z) + \max(x;y;z) \geq 0$, prove that

$$E(x;y;z) \leq E(\min(x;y;z);\min(x;y;z);\max(x;y;z))$$

b). If $x, y, z \in R$ so that $x \cdot y \cdot z > 0$, $\min(x;y;z) < 0$ and $\min(x;y;z) + \max(x;y;z) \geq 0$ prove that $E(x;y;z) \leq 1$.

Ovidiu Pop

W29. Let $m \in N^*$, $I \subset R$, I interval, $f : I \to R$ be a function, m times differentiable on I and the distinct knots $x_0, x_1, ..., x_m \in I$. Prove that $f \in I$ exists so that

$$[x_0, x_1, ..., x_m; f]\left(1 + (m+1)\zeta - \frac{x_0 + x_1 + ... + x_m}{V(x_0, x_1, ..., x_m)}\right) = \frac{1}{m!} f^{(m)}(\zeta),$$

where $V(x_0, x_1, ..., x_m)$ is the Vandermonde determinant and $[x_0, x_1, ..., x_m; f]$ is the divided difference of the function f on the knots $x_0, x_1, ..., x_n$.

Ovidiu Pop

W30. Let $x \in Poisson\ (2)$ be a random variable. Find all the values $n \in N^*$ so that:

$$P\left(\left\{|\omega|\, |x(\omega) - 2| \geq \frac{2}{n}\right\}\right) \leq \frac{128}{n^2}$$

Laurenţiu Modan

W31. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(1 + \frac{1}{n}\sum_{k=1}^{n} a_k\right)^{\frac{n}{\sum\limits_{k=1}^{n} a_k}} \leq \left(1 + \sqrt[n]{\prod_{k=1}^{n} a_k}\right)^{\frac{1}{\sqrt[n]{\prod\limits_{k=1}^{n} a_k}}} \leq \left(1 + \frac{n}{\sum\limits_{k=1}^{n} \frac{1}{a_k}}\right)^{\frac{1}{n}\sum\limits_{k=1}^{n} \frac{1}{a_k}}$$

Mihály Bencze

W32. If $0 < a \leq b$ then

$$\ln \frac{b(2a+\pi)}{a(2b+\pi)} < \int_a^b \frac{arctgx}{x^2} dx < \frac{\pi}{2} \ln \frac{b(a\pi+2)}{a(b\pi+2)}$$

Mihály Bencze

W33. If $x_i \geq 1$ $(i = 1, 2, ..., n)$ and $k \in N^*$, then

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \geq \sum_{cyclic} \frac{1}{1 + \sqrt[n+k-1]{x_1^k x_2 x_3 ... x_n}}$$

Mihály Bencze

W34. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N^*$, then

$$\sum_{i=1}^{n} \frac{a_i^{k-1}}{(1+a_k)^{n(k-1)+\frac{k(k+1)}{2}-1}} \le \frac{n(1+a_1)^2(1+a_2)^3...(1+a_n)^{n+1}}{(n+k)^{n+k} a_1 a_2 ... a_n}$$

Mihály Bencze

W35. If $x_i \ge 1$ $(i = 1, 2, ..., n)$ and $k \in N^*$, then

$$\sum_{cyclic} \left(x_1^k x_2 ... x_n - \frac{1}{x_1^k x_2 ... x_n} \right) \ge (n+k-1) \sum_{i=1}^{n} \left(x_i - \frac{1}{x_i} \right)$$

Mihály Bencze

W36. If $a_i > 0$ $(i = 1, 2, ..., n)$, $k \in N$, $k \ge 2$ then

1). $\sum\limits_{cyclic} \frac{1}{a_1+a_2+...+a_{n-1}} \ge \frac{k\sum\limits_{i=1}^{n} a_i^{k-1}}{\sqrt[k]{(k-1)^{k-1}(n-1)^{k-1}} \sum\limits_{i=1}^{n} a_i^k}$

2). $\sum\limits_{cyclic} \frac{1}{a_1^{k-1}+a_2^{k-1}+...+a_{n-1}^{k-1}} \ge \frac{k\sum\limits_{i=1}^{n} a_i}{\sqrt[k]{(k-1)^{k-1}(n-1)} \sum\limits_{i=1}^{n} a_i^k}$

Mihály Bencze

W37. If $a, b, c > 0$ then

$$\sum \left(a^4 + 3b^4 + 3a^2c^2 + 3b^2c^2\right) \sqrt{\frac{a^3}{a^3 + (b+c)^3}} \ge \left(\sum a^2\right)^2$$

Mihály Bencze

W38. If $n, k \in N^*$, then

$$k\left(1 + \frac{1}{2} + ... + \frac{1}{n}\right) + (k-1)\left(\frac{1}{n+1} + ... + \frac{1}{n^2}\right) + (k-2)\left(\frac{1}{n^2+1} + ... + \frac{1}{n^3}\right) + ...$$

$$+2\left(\frac{1}{n^{k-2}+1} + ... + \frac{1}{n^{k-1}}\right) + \left(\frac{1}{n^{k-1}+1} + ... + \frac{1}{n^k}\right) \ge$$

$$\ge \frac{k(k+1)}{2} \ln n + \frac{1}{2}\left(k + \frac{n^k - 1}{(n-1)n^k}\right)$$

Mihály Bencze

W39. Prove that

$$\frac{n(n+1)(n+2)}{3} < \sum_{k=1}^{n} \frac{1}{\ln^2\left(1+\frac{1}{k}\right)} < \frac{n}{4} + \frac{n(n+1)(n+2)}{3}$$

Mihály Bencze

W40. Prove that

$$\sum_{k=1}^{n} \frac{\ln\left(1+\frac{1}{k}\right)}{2k+1} < \frac{n}{n+1} < 2\sum_{k=1}^{n} \frac{1}{(2k+1)\ln(2k+1)}$$

Mihály Bencze

W41. Let be $x_0 = 0, x_1 = 1$ and $x_{n+2} = (2n+5)\, x_{n+1} - \left(n^2 + 4n + 3\right) x_n$ for all $n \in N$. Prove that:
1). $x_n \in N$ for all $n \in N$
2). x_{4n} is divisible by $n\,(4n)!$
3). x_{4n+1} is divisible by $(n+1)\,(4n+1)!$

György Szöllősy