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THE SCIENTIFIC WORK OF VITO VOLTERRA*

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Death came last October to one of the most renowned, and most deservedly renowned, mathematicians of our time—Vito Volterra. Looking at his accomplishments for a few minutes, we can sense pride in the possibilities of this human race, and remorse that the performance of most of us falls so far short of it.

Born at Ancona in 1860, Volterra began his university study at Florence, but soon went to Pisa; there his doctorate was granted by the Scuola Normale Superiore in 1882. At that school, winning a competition for the chair of rational mechanics, he entered his academic career in 1883. Turin called him to a similar post in 1892, and from 1900 to 1931 Volterra was Professor of Mathematical Physics at Rome.

Membership in academies, doctorates of universities in many countries testify amply of the esteem in which he was held. He served the international scientific world as President of the International Commission on Weights and Measures. The Italian nation benefited by his work and thought in the political field as Senator, in the scientific field as chief reviver of the Italian Society for Progress of Science. Among the universities which invited him to give series of lectures, and then found them worthy of publication, were the University of California, Rice Institute, and Clark University in this country, the Universities of Brünn and Prague, Madrid and Paris, Stockholm and Upsala in Europe.

A great scientist and teacher will be known not only by his work but also by the distinguished students whom he attracts and trains. A few who owe much to the man whose memory we are honoring are Evans, Pérès, Brelot, Fantappiè, and Tonelli.

Once Volterra wrote: "Men of science love to study the treasures gathered by others, in order to know well the value of their own. In the person dedicated to mathematical studies such curiosity is much greater than in those occupied with other disciplines." He himself certainly came to confirm this boast. For while his chief gifts to science lie in pure mathematics, there was a continual interchange of stimulus between mathematics and its application to mechanics, physics and chemistry, to aeronautics, to geology, to biology and economics. A particular problem would set him at work leading to a mathematical theory; the theory itself suggested other fields for its application.

In mathematics Volterra was unusually resourceful in the method so often fruitful in mathematician's hands, of proceeding from few to many, from finite to infinite.

Volterra's first published paper was written before he was twenty. It treated of point-wise discontinuous functions.† Soon there followed papers on a gen-

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† "If a point-wise discontinuous function f has points of discontinuity in every sub-interval of its interval of definition, there exists no other function discontinuous where f is continuous and continuous where f is discontinuous" was the first theorem published.

eralization of Riemann integration, and on conditions determining a function of a complex variable. This latter, to be sure, drew rather sharp criticism from Schwarz.

Volterra's main attention at this time, however, was given to potential theory. It was the subject of his habilitation thesis, and found applications to electrochemistry (deposits on a cylinder in an electrolyte), electrostatics, thermodynamics, hydrodynamics. Much later this interest in potential resulted in a device for predicting the distribution of temperature in the interior of a mountain—through the equipotential lines on a metallic model.

Differential equations appear in the titles of many papers. In the first series of such notes (1887) these equations appear in the title but are neglected in the text. The fact that integrals of linear differential equations with algebraic coefficients undergo substitutions on the variables making certain circuits led to a study of linear substitutions (or matrices), with definition of their differentials, derivatives, and integrals. The elements of the substitutions were next allowed to depend on a complex variable (so that residues of substitutions could be brought in) or on a point of a Riemann surface. A later section of this study was published in 1899, when the idea of the functional had been developed and could be applied to integrals of substitutions.

Perhaps the most important concept which we owe to Volterra is that of functional, a number dependent, not on another number, but on all values of one or more functions—or on a closed curve. The theory was established in 1887. The derivative and variation of functions of functions, the directional derivatives of functions of closed curves were studied at once; and then came complex functionals, functionals in hyperspaces. An analog of the Taylor development is now generally known as the Volterra series. And an extension of Green's theorem brought this interesting result: if a function of a line is known for every closed curve on the boundary of a domain, it is fixed for every closed curve of the interior.

Interest in the calculus of variations naturally accompanied this development of functions. For whereas more elementary problems of maxima and minima call for extremal values of functions of one variable or of a finite number, the calculus of variations would, for instance, maximize a function of a line by the proper choice of that line. One fruit of this interest was his deduction of the Hertz electromagnetic equations from a variational principle.

Then he turned to the study of a mechanical system having internal motion, such as the earth with its liquid core. There were ten papers on this topic in 1895 alone. Particular attention was given to the resulting motion of the pole. This series of studies was probably a chief reason for their author's last honor; in January 1940 the Pontifical Academy appointed him on an international commission to study the age of the earth. It was reported that the Vatican's pleasure in thus honoring Volterra was increased by its underlining the displeasure of the church with Mussolini's imported anti-Semitism.

Of course Volterra is best known as an investigator of integral equations. As early as 1884 he published a note, relating to the distribution of electricity on a zone of a sphere, which involved the solution of an integral equation of the first kind. But it was in 1896 that his systematic treatment of integral equations was first printed. Half a dozen papers of that year discuss iterated kernels, the resolvent kernel, its use in the solution of an equation of the second kind, the reduction of one of the first kind to one of the second, and the extension of the results to multiple integrals and to systems of equations.

Interest in the composition of functions—the process used to obtain iterated kernels—led, some fifteen years later, to further detailed study concerning permutable functions.

Integro-differential equations began to appear among his writings by 1909, and the many articles on them cover several years. They are often connected with hereditary phenomena, phenomena in which the future behavior of a system depends not only on its present condition (as in classical mechanics) but also on its past history. This is suggested, in the first place, by the effect of protracted strain on the elastic properties of matter; then by hysteresis in electromagnetism—here Volterra generalized the Hertz equations and obtained an integro-differential equation to characterize periodicity. Economics is another field wherein hereditary phenomena enter—the memory of past prices of wheat has a definite influence on the price which can be demanded at a particular time.

During the war from 1914 to 1918 Volterra was occupied with the theoretical and experimental study of dirigibles. I have, however, found no record of the results of this research.

Soon after the war he took up the application of mathematics to biology, in particular to the competition of species. On the hypothesis that some are prey to others, differential equations were set up for the populations of the diverse varieties. These assumed the effect of any ingestion to be immediate. When, however, allowance was made for the time needed for reproduction, the delay gave rise to integro-differential equations. The last phase of Volterra's study of the struggle for existence was the development of a vital mechanics, a mechanics which allowed the equations to be deduced from a variational principle. A sample theorem may be of interest: "If there is an isochrone modification of the natural passage from one state of a biological association to another, such that the quantities of life at initial and final instants are unaltered and the virtual work of growth is zero at each instant, then the vital action must increase."

Early in this century Vito Volterra gave particular attention to the problems of elasticity. Discovering that an elastic body can be in a state of stress without the presence of mass force or surface tension if, and only if, it occupies a multiply connected space, he made both theoretical and experimental studies of such bodies. A typical theorem is this: "If, in a multiply connected body, two systems of distortions correspond to two systems of forces, the sum of products of forces of the first system by corresponding strains of the second equals that of strains of the first by forces of the second." Part of the experimental work was done with gelatine models, in which the optics of doubly refracting media (another topic studied by Volterra) permitted the visualization of stresses.

In closing I would pay tribute to the integrity of this scientist. At the time when the fascist oath was exacted of all teachers, Volterra realized that a promise of allegiance to the regime of Mussolini would violate the pledge he had made on becoming Senator of the kingdom. With sorrow he left the University which he had so greatly served and honored.

THE FACTORIZATION OF CERTAIN SECOND ORDER POLYNOMIAL DIFFERENTIAL OPERATORS

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1. Introduction. We shall consider differential operators which are polynomials in a variable x and in $D\equiv d/dx$. We are concerned with the factorization, if possible, of such operators into factors which are operators of the same type. Our aim is the actual determination of such factors, not the exhibiting of criteria for reducibility. The problem is solved here, in the above sense, only for a restricted set of polynomial differential operators. A previous paper on Riccati equations furnishes the tool for our present attack. It will be seen that one reasonable line of approach to the general problem is an extension of the results of that paper.

In the study of the algebraic properties of linear differential operators, Blumberg's thesis* may well be used as a starting-point. Most writers on the reducibility of such operators concern themselves with coefficients which are rational† functions of x. The results are naturally vastly different from those in the problem considered here. F. H. Miller‡ has discussed a variation of reducibility in which the coefficients of the operator are also rational in a parameter.

With the understanding that in this paper we speak only of polynomial differential operators, we shall use the terms divisor, proper divisor, and trivial operator or divisor in the usual sense. If A, B, C, are operators such that $A=BC$, then B and C are *divisors* of A and A is a *multiple* of B and of C. At times we are more precise and speak of right and left divisors and multiples. For instance, B is a *left divisor* of A and A is a *right multiple* of B. An operator, or divisor, is said to be *trivial* if, and only if, it is independent of both x and D. A *proper divisor* of an operator is a divisor which is neither trivial nor a trivial multiple of the original operator. Again we may speak of right and of left proper divisors.

2. A class of second order operators. Consider the operator

$$y=D^2+a_1(x)D+a_0(x),$$

where a_1 and a_0 are polynomials in x. If y has a proper divisor, then evidently y may be written

* H. Blumberg, Über algebraische Eigenschaften von linearen homogenen Differentialausdrücken, Göttingen, 1912.

† See remarks on page 9 of F. H. Miller's thesis, Reducible and irreducible linear differential equations, Columbia, 1932.

‡ See Miller, *loc. cit.*, also for further references on the question of reducibility.