Secondo minicompito

❶

$$I = \int_0^1 \ln\left(\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1}\right)\frac{dx}{x} = \int_0^1 \frac{\ln\left(x^2+\sqrt{3}x+1\right)}{x}dx - \int_0^1 \frac{\ln\left(x^2-\sqrt{3}x+1\right)}{x}dx$$

$$\bar{I}(a) = \int_0^1 \frac{\ln(x^2+2x\cos(a)+1)}{x}dx$$

$$\overline{I'}(a) = \int_0^1 \frac{\partial}{\partial a}\frac{\ln(x^2+2x\cos(a)+1)}{x}dx = -\int_0^1 \frac{2\sin(a)}{x^2+2x\cos(a)+1}dx \xrightarrow{t=\frac{x+\cos(a)}{\sin(a)}}$$

$$-\int_{\frac{\cos(a)}{\sin(a)}}^{\frac{1+\cos(a)}{\sin(a)}} \frac{2\sin^2(a)}{(t\sin(a)-\cos(a))^2+2(t\sin(a)-\cos(a))\cos(a)+1}dt$$

$$= -\int_{\cot(a)}^{\frac{2\cos^2\left(\frac{a}{2}\right)}{2\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right)}} \frac{2\sin^2(a)}{t^2\sin^2(a)-\cos^2(a)+1}dt = -\int_{\cot(a)}^{\cot\left(\frac{a}{2}\right)}\frac{2}{t^2+1}dt$$

$$= -2\left(\arctan\left(\cot\left(\frac{a}{2}\right)\right)-\arctan(\cot(a))\right)$$

$$= -2\left(\frac{\pi}{2}\operatorname{sign}(a)-\frac{a}{2}-\frac{\pi}{2}\operatorname{sign}(a)+a\right) = -a, \qquad \forall a \in (-\pi,\pi)$$

$$I = \bar{I}\left(\frac{\pi}{6}\right)-\bar{I}\left(\frac{5}{6}\pi\right) = \int_{\frac{5}{6}\pi}^{\frac{\pi}{6}} -a\,da = \frac{\left(5\frac{\pi}{6}\right)^2-\left(\frac{\pi}{6}\right)^2}{2} = \frac{25\pi^2-\pi^2}{72} = \frac{\pi^2}{3}$$

❷

$$① \; z\coth(z) = 1 + \sum_{k\in\mathbb{N}} \frac{B_{2k}(2z)^{2k}}{(2k)!}$$

$$② \; \zeta(2k) = \frac{(-1)^{k+1}B_{2k}(2\pi)^{2k}}{2(2k)!}$$

$$③ \; \sum_{n\in\mathbb{N}} \frac{a}{n^2+a^2} = a\sum_{n\in\mathbb{N}} \frac{1}{n^2}\cdot\frac{1}{1+\frac{a^2}{n^2}} = a\sum_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{a^2}{n^2}\right)^k = a\sum_{k=0}^{\infty}(-1)^k a^{2k}\sum_{n\in\mathbb{N}}\frac{1}{n^{2k+2}}$$
$$= a\sum_{k=0}^{\infty}(-1)^k a^{2k}\zeta(2k+2) = a\sum_{k\in\mathbb{N}}(-1)^{k-1}a^{2k-2}\zeta(2k) \overset{②}{\to} \frac{1}{2a}\sum_{k\in\mathbb{N}}\frac{B_{2k}(2\pi a)^{2k}}{(2k)!} \overset{①}{\to}$$
$$\to \frac{\pi}{2}\coth(\pi a) - \frac{1}{2a}$$

$$④ \; \sum_{n\in\mathbb{N}}\frac{a}{(2\pi n)^2+a^2} \xrightarrow{a=2\pi b} \sum_{n\in\mathbb{N}}\frac{2\pi b}{(2\pi n)^2+(2\pi b)^2} = \frac{1}{2\pi}\sum_{n\in\mathbb{N}}\frac{b}{n^2+b^2} \overset{③}{\to} \frac{1}{4}\coth(\pi b) - \frac{1}{4\pi b} \xrightarrow{b=\frac{a}{2\pi}}$$
$$\to \frac{1}{4}\coth\left(\frac{a}{2}\right) - \frac{1}{2a} = \frac{1}{4}\left(\coth\left(\frac{a}{2}\right) - \frac{2}{a}\right)$$

$$J(a) = \int_0^\infty \frac{\sin(ax)}{e^{2\pi x}-1}\,dx = \int_0^\infty \sin(ax)\frac{e^{-2\pi x}}{1-e^{-2\pi x}}\,dx = \int_0^\infty \sin(ax)\sum_{n=0}^{\infty} e^{-2\pi(n+1)x}\,dx$$
$$= \sum_{n\in\mathbb{N}}\int_0^\infty e^{-2\pi n x}\sin(ax)\,dx = \sum_{n\in\mathbb{N}}\mathcal{L}_x\{\sin(ax)\}(2\pi n) = \sum_{n\in\mathbb{N}}\frac{a}{(2\pi n)^2+a^2} \xrightarrow{④}$$
$$\to \frac{1}{4}\left(\coth\left(\frac{a}{2}\right) - \frac{2}{a}\right)$$