ROME-MOSCOW SCHOOL 2016

COURSE: SELF-SIMILAR ENERGIES ON FRACTALS
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Problems

1) Let A be the line-segment in $\mathbf{R}^2$ with end-points $(1,-1)$ and $(-1,1)$, and let B_r be the circle with center at the origin and radius $r>0$, in formula

$$B_r = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = r^2\}.$$

Compute, for every $r>0$, $d_H(A, B_r)$ where d_H denotes the Hausdorff distance and $\mathbf{R}^2$ is equipped with the usual euclidean distance.

2) Consider the Sierpinski Gasket (short S.G.), let P_1, P_2, P_3 be as usually the vertices of the triangle, i.e., the fixed points of the contractions. Find $u : V^{(0)} \to \mathbf{R}$ such that, if v denotes $H_{1,E}(u)$, we have $v(P_2) = 0$, $v(P_{12}) = 2$, $v(P_{23}) = 3$. Here, as usual,
$P_{12} = \psi_1(P_2) = \psi_2(P_1)$, $P_{23} = \psi_2(P_3) = \psi_3(P_2)$.

3) Prove that in S.G. there exists a function $v : V^{(1)} \to \mathbf{R}$ such that
$v(P_1) = 0$, $v(P_2) = 1$, $v(P_3) = 2$, and $S_1(\overline{E})(v) = 10^{30}$.
Here, $\overline{E}(u) = \big(u(P_2) - u(P_1)\big)^2 + \big(u(P_3) - u(P_1)\big)^2 + \big(u(P_3) - u(P_2)\big)^2$. Recall that

$$S_1(E)(v) = \sum_{i=1}^{k} E(v \circ \psi_i)$$

where $k = 3$ in the case of the Gasket.

4) Let $V^{(0)}$ be a set with at least two elements,

$$V^{(0)} = \big\{P_1, P_2, ..., P_N\big\}, \quad N \geq 2.$$

Let $E \in \mathcal{D}$. Recall that this means that E is a form for functions from $V^{(0)}$ to $\mathbf{R}$ of the type

$$E(u) = \sum_{1 \leq j_1 < j_2 \leq N} c_{j_1,j_2}(E)\big(u(P_{j_2}) - u(P_{j_2})\big)^2$$

where $c_{j_1,j_2}(E)$ are the *coefficients* of E, and they are required to be non-negative and we can assume $c_{j_1,j_2}(E) = c_{j_2,j_1}(E)$ for $j_1 \neq j_2$.
Determine for what $N \geq 3$ the form E only depends on the values $E(\chi_{\{P_j\}})$ at the functions of the form $\chi_{\{P_j\}}$, $j = 1, ..., N$, where χ denotes the characteristic function.
The phrase *only depends on the values* $E(\chi_{\{P_j\}})$ of course means that, if $E_1, E_2 \in \mathcal{D}$ and $E_1(\chi_{\{P_j\}}) = E_2(\chi_{\{P_j\}})$ fo every $j = 1, ..., N$, then $E_1 = E_2$.
NOTE: I exclude the case $N = 2$ since it is trivial.

5) (Difficult) Suppose we are in the class of fractals considered in the course. Suppose E is an eigenform with positive coefficients and eigenvalue $\rho > 0$, that is, $E \in \mathcal{D}$, $c_{j_1,j_2}(E) > 0$ for all $j_1, j_2 = 1, ..., N$ with $j_1 \neq j_2$, and $\Lambda(E) = \rho E$. Let, as usual $T_{j,E}(u) = H_{1,E}(u) \circ \psi_j$ for $j = 1, ..., N$ and

$$u \in \mathbf{R}_j^{V^{(0)}} =: \left\{ u \in \mathbf{R}^{V^{(0)}} : u(P_j) = 0 \right\}$$

Recall that $T_{j,E} : \mathbf{R}_j^{V^{(0)}} \to \mathbf{R}_j^{V^{(0)}}$ is a positive linear operator. Let l_j be its positive eigenvalue given by the Perron-Frobenius theorem.

Prove that $l_j = \rho$.