4. Analysis on Fractals

4.1 Renormalization Operator.

In this chapter we will investigate analysis on fractals. A large part of it is given in [P], thus here we will only discuss the parts not discussed there. We will often refer to [P] for the notation and result. However, we will recall some basic notation. We recall that we are given a self-similar set Γ generated by finitely many contracting similarities ψ_i, $i = 1, ..., k$, as stated in Section 3.3. We also assume O.S.C. holds. Let P_i be the fixed point of ψ_i. We call $V^{(0)}$ a subset of the set of the fixed points $\{P_1, ..., P_k\}$. By possibly changing the indices we can assume $V = V^{(0)} := \{P_1, ..., P_N\}$, and we suppose $2 \le N \le k$. Usually, the set $V^{(0)}$ is not arbitrary, but it is defined in an exact way. We can define it as the set of *essential* fixed points, where a fixed point P_j is essential if there exists $j' \ne j,\ i, i' = 1, ..., k$ such that $\psi_i(P_j) = \psi_{i'}(P_{j'})$. Another way is to define $V^{(0)}$ geometrically as the extrema of $\overline{\hat{A}}$, where $\hat{A}$ is the open set given by O.S.C., that is $\Gamma = co(V^{(0)})$. This is essentially the definition given in [P]. Since in the most of cases, such definitions are equivalent, we will not investigate further this point. We will define *n-cell* a set of the form $V_{i_i,...,i_n}$, and put $V^{(n)} = \bigcup_{i_1,...,i_n=1}^{k} V_{i_i,...,i_n}$. Note that $V (= V_\emptyset)$ is the unique 0-cell. Recall that $V^{(n)}$ is an increasing sequence of sets and $\Gamma = \overline{\bigcup_{n=0}^{+\infty} V^{(n)}}$. In the fractals we will consider, we require the following properties:

a) $P_j \notin V_i$ when $i \ne j$
b) $V^{(1)}$ is connected in the sense that any two points in $V^{(1)}$ can be connected by a path whose any edge belongs to a 1-cell, which, of course depends on the edge.
c) If $i_1, ..., i_n, i'_1, ..., i'_n = 1, ..., k$ and $(i_1, ..., i_n) \ne (i'_1, ..., i'_n)$, then $V_{i_1,...,i_n} \ne V_{i'_1,...,i'_n}$, and $V_{i_1,...,i_n} \cap V_{i'_1,...,i'_n} = \Gamma_{i_1,...,i_n} \cap \Gamma_{i'_1,...,i'_n}$.
Property c) is called nesting axiom or finite ramification. See [P] for more details. A *nested fractal* is a self-similar set with the above property which, furthermore, satisfies the following symmetry property.

d) The contraction factors σ_i are all equal. Moreover, if $P, Q \in V$, $P \ne Q$, then the symmetry $S_{P,Q}$ with respect to $H(P,Q) = \{z : ||z - P|| = ||z - Q||\}$, maps n-cells to n-cells for $n \ge 0$, and any n-cell containing elements on both sides of $H(P,Q)$ is mapped to itself.

Examples of nested fractals are the Sierpinski Gasket, the Vicsek set, and the Lindstrøm Snowflake. As stated above, we will always require properties a), b), c), not necessarily d). We now recall the definition of the renormalization operator Λ_r. For the definition of $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ see [P]. For every $u \in \mathbb{R}^{V^{(0)}}$, for every $E \in \widetilde{\mathcal{D}}$ and for every $r : \{1, ..., k\} \to]0, +\infty[$, let

$$\Lambda_r(E)(u) = \inf\Big\{\sum_{i=1}^{k} r_i E(v \circ \psi_i) : v \in \mathcal{L}(u)\Big\}, \quad \mathcal{L}(u) = \big\{v \in \mathbb{R}^{V^{(1)}}, v = u \text{ on } V^{(0)}\big\}.$$

It is well known that the infimum is attained at a unique function. In the following, we will put $\Lambda := \Lambda_1$, where 1 denotes r such that $r_i = 1$ for all i, and we will usually only consider this case. Note that in [P], a different notation (M_1) is used to denote the renormalization operator. Recall that $E \in \widetilde{\mathcal{D}}$ is said to be an eigenform if there exists $\rho > 0$ such that $\Lambda(E) = \rho E$, in other words an eigenform is an eigenvector of the (nonlinear) operator Λ.

4.2 Nested Fractals

From now on, we will assume that the fractal considered is a nested fractal. Although it is possibly not strictly necessary we will assume further

e) $\#(V_{i_1} \cap V_{i_2}) \leq 1$ *for every* $i_1, i_2 = 1, ..., k$ *with* $i_1 \neq i_2$.

Note that e) is satisfied by all nested fractals mentioned here, and, to my knowledge, there are no known examples of nested fractals not satisfying e).

Remark 4.2.1. We note that the symmetry property of nested fractals (d) in previous section) implies for example that, fixing a positive number $\bar{d}$, every vertex P_j has the same number of vertices having distance $\bar{d}$ from it. In fact, given $P_j, P_{j'} \in V^{(0)}$, the symmetry $S_{P_j,P_{j'}}$ maps $V^{(0)}$ into itself by d), as it contains the points P_j and $P_{j'}$ on different sides of $H(P_j, P_{j'})$, so that, with every vertex P_h such that $d(P_j, P_h) = \bar{d}$ there exists a vertex $S_{P_j,P_{j'}}(P_h)$ and

$$d(P_{j'}, S_{P_j,P_{j'}}(P_h)) = d\big(S_{P_j,P_{j'}}(P_j), S_{P_j,P_{j'}}(P_h)\big) = d(P_j, P_h) = \bar{d}. \qquad \blacksquare$$

Let $d_i < d_2 < \cdots < d_m$ be the distance between different elements of $V^{(0)}$, and $m_i = \#\{P \in V^{(0)} : d(P, P_j) = d_i\}$. In such a definition P_j is any element of $V^{(0)}$, and the definition makes sense as by Remark 4.2.1, such a number m_i is independent of j. We define $\mathcal{G}_{min}$ to be the graph on $V^{(0)}$ whose edges are $\{P_{j_1}, P_{j_2}\}$ such that $d(P_{j_1}, P_{j_2}) = d_1$, in other words the different points of minimum distance. For notions about graphs see [P].

Lemma 4.2.2. *The graph $\mathcal{G}_{min}$ is connected.*

Proof. Note that by the definition of the symmetry $S_{P,Q}$, we have

$$d(Q_1, Q_2) \geq d(Q_1, H(P, Q)) + d(Q_2, H(P, Q)) \tag{4.2.1}$$

if Q_1 and Q_2 lie in opposite sides of $H(P, Q)$, and

$$d(Q_1, Q_2) \geq |d(Q_1, H(P, Q)) - d(Q_2, H(P, Q))|$$

if Q_1 and Q_2 lie in the same side of $H(P, Q)$. Moreover, if Q_1 and Q_2 lie in opposite sides of $H(P, Q)$ and $d(Q_1, H(P, Q)) = d(Q_2, H(P, Q))$, then the equality $d(Q_1, Q_2) = 2d(Q_1, H(P, Q))$ holds if and only if $Q_2 = S_{P,Q}(Q_1)$.

Suppose now the Lemma is false. Let

$$A := \left\{P \in V^{(0)} : P \text{ is connected to } P_1\right\},$$

$$B := \left\{P \in V^{(0)} : P \text{ is not connected to } P_1\right\},$$

where by connected we mean connected in $\mathcal{G}_{min}$. The set A is nonempty, as it contains P_1, and the set B is nonempty as, if all elements of $\mathcal{G}_{min}$ would be connected to P_1, then the graph $\mathcal{G}_{min}$ would be connected, contrary to our assumption. Of course, any two elements of A are connected in $\mathcal{G}_{min}$, any two elements of B are connected in $\mathcal{G}_{min}$, and an element of A is not connected in $\mathcal{G}_{min}$ to an element of B. There exist $P_{j_1} \in A$ and $P_{j_2} \in B$ of minimum distance, and, clearly $d(P_{j_1}, P_{j_2}) > d_1$, as if $d(P_{j_1}, P_{j_2}) = d_1$, then P_{j_1} is connected to P_{j_2}. Let $P \in V^{(0)}$ be such that $d(P, P_{j_1}) = d_1$. Then, P is connected to P_{j_1}, thus is in A, and $d(P, P_{j_2}) \geq d(P_{j_1}, P_{j_2})$ by the definition of P_{j_1}, P_{j_2}. Note that P_{j_1} and P lie in the same side of H, as, otherwise, $d(P_{j_1}, P_{j_2}) < d(P_{j_1}, P) = d_1$, a contradiction. Let $S := S_{P,P_{j_2}}$, $H := H_{P,P_{j_2}}$. We have

$$2d(P_{j_2}, H) = d(P, P_{j_2}) \geq d(P_{j_1}, P_{j_2}) \geq d(P_{j_2}, H) + d(P_{j_1}, H) \tag{4.2.2}$$

hence $d(P_{j_1}, H) \leq d(P_{j_2}, H)$, and in fact the strict inequality holds, as, if $d(P_{j_1}, H) = d(P_{j_2}, H)$, then the inequalities in (4.2.2) are in fact equalities, in particular, $d(P_{j_1}, P_{j_2}) = d(P_{j_2}, H) + d(P_{j_1}, H)$, which implies $P_{j_1} = S(P_{j_2})$, but by definition of S, $S(P_{j_2}) = P$, a contradiction. Let now $\bar{P} := S(P_{j_1})$. As $d(\bar{P}, P_{j_2}) = d(S(P_{j_1}), S(P)) = d(P_{j_1}, P) = d_1$, then $\bar{P}$ is connected in $\mathcal{G}_{min}$ to P_{j_2}, thus $\bar{P} \in B$. Moreover,

$$d(P_{j_1}, \bar{P}) = 2d(P_{j_1}, H) < d(P_{j_2}, H) + d(P_{j_1}, H) \leq d(P_{j_1}, P_{j_2})$$

and as $P_{j_1} \in A$ and $\bar{P} \in B$, this contradicts the definition of P_{j_1}, P_{j_2}. ∎

As every form $E \in \mathcal{D}$ is uniquely determined by its coefficients, we can in some sense identify E with the set of its coefficients The coefficients are objects of the form c_{j_1,j_2} with $(j_1, j_2) \in \{1, ..., N\} \times \{1, ..., N\}, j_1 \neq j_2$. We thus have $N^2 - N = N(N-1)$ coefficients, and the set of the coefficients, being a set of $N(N-1)$ real numbers, can be identified with an element of $\mathbb{R}^{N(N-1)}$. In the following we will denote by $\mathcal{H}$ the set of $c \in \mathbb{R}^{N(N-1)}$ such that

i) $c_{j_1,j_2} = c_{j_2,j_1} \geq 0$,
ii) $\sum\limits_{j_2 \neq j} c_{j,j_2} = 1$ for all $j = 1, ..., N$,
iii) $c_{j_1,j_2} \leq c_{j_3,j_4}$ when $||P_{j_1} - P_{j_2}|| \geq ||P_{j_3} - P_{j_4}||$.

Property ii) is in some sense a normalization property. Property iii) states that the coefficients are decreasing with respect to the distance, in particular they only depend on the distance (that is, if $||P_{j_1} - P_{j_2}|| = ||P_{j_3} - P_{j_4}||$, then $c_{j_1,j_2} = c_{j_3,j_4}$). Then, using Remark 4.2.1, by iii), in order that ii) holds it suffices it holds for one $j = 1, ..., N$.

Lemma 4.2.3 *Every E whose set of coefficients lies in $\mathcal{H}$, is in $\widetilde{\mathcal{D}}$.*

Proof. By iii) the biggest coefficient c_{j_1,j_2} is when $||P_{j_1} - P_{j_2}|| = d_1$ and by ii) such a coefficient is strictly positive. By Lemma 4.2.2, the graph on $V^{(0)}$ whose edges are $\{P_{j_1}, P_{j_2}\}$ such that $c_{j_1,j_2} > 0$ is connected, and this means that $E \in \widetilde{\mathcal{D}}$ (see [P]). ∎

Remark 4.2.4. Note that the coefficients $c_{j_1,j_2}(\Lambda(E))$ of $\Lambda(E)$ continuously depend on those of E. Recall (see [P]) that

$$c_{j_1,j_2}(\Lambda(E)) = \frac{1}{4}\Big(\Lambda(E)(\chi_{P_{j_1}} - \chi_{P_{j_2}}) - \Lambda(E)(\chi_{P_{j_1}} + \chi_{P_{j_2}})\Big)\,. \tag{4.2.3}$$

Moreover, note that $\Lambda(E)(u) = S_1(H_{1;E}(u))$, and $H_{1;E}(u)$ is the solution of a linear system in $\#(V^{(1)})$ unknown variables of the kind $Ax = b$ where the entries $a_{i,j}$ of the matrix A and the components b_i of b are linear combinations of $c_{j_1,j_2}(E)$ and $u(P_j)$ (see [P] again). The solution to such a system is the quotient of two determinants which are polynomial, thus continuous functions of $a_{i,j}$ and b_i, therefore continuous functions of $c_{j_1,j_2}(E)$, then $H_{1;E}(u)$ continuously depend on $c_{j_1,j_2}(E)$. Using $u = \chi_{P_{j_1}} \pm \chi_{P_{j_2}}$, in view of (4.2.3), we thus get that the coefficients of $\Lambda(E)$ continuously depend on $c_{j_1,j_2}(E)$, as claimed. $\blacksquare$

It is easy to see that $\mathcal{H}$ is nonempty, compact and convex, so that we will get the existence of an eigenform by proving that the continuous map $E \mapsto \Lambda(E)$, normalized, dividing it by a suitable quantity depending on E, takes values into $\mathcal{H}$, therefore it does have a fixed point.

Remark 4.2.5. If the set of coefficients of $E \in \widetilde{\mathcal{D}}$ lies in $\mathcal{H}$, then the coefficients $c_{j_1,j_2}(\Lambda(E))$ only depend on the distance $||P_{j_1} - P_{j_2}||$. We will hint the proof of this fact, which is a symmetry argument. Suppose first $||P_{j_1} - P_{j_2}|| = ||P_{j_1} - P_{j_3}||$, and prove that $c_{j_1,j_2}\big(\Lambda(E)\big) = c_{j_1,j_3}\big(\Lambda(E)\big)$. The symmetry $S_{P_{j_2},P_{j_3}}$ leaves P_{j_1} fixed, exchanges P_{j_2} and P_{j_3}, maps 1-cells into 1-cells. Therefore, $S_{P_{j_2},P_{j_3}}$ sends $H_{1;E}(u)$ into $H_{1;E}(u \circ S_{P_{j_2},P_{j_3}})$. As the coefficients of E only depend on the distance by the hypothesis iii), we have

$$\Lambda(E)(u) = S_1(E)\Big(H_{1;E}(u)\Big) = S_1(E)\Big(H_{1;E}\big(u \circ S_{P_{j_2},P_{j_3}}\big)\Big) = \Lambda(E)\big(u \circ S_{P_{j_2},P_{j_3}}\big)$$

As $\chi_{P_{j_1}} \circ S_{P_{j_2},P_{j_3}} = \chi_{P_{j_1}}$, and $\chi_{P_{j_2}} \circ S_{P_{j_2},P_{j_3}} = \chi_{P_{j_3}}$, by (4.2.3) we have $c_{j_1,j_2}\big(\Lambda(E)\big) = c_{j_1,j_3}\big(\Lambda(E)\big)$, as claimed. Suppose now,

$$||P_{j_1} - P_{j_2}|| = ||P_{j_4} - P_{j_3}|| \tag{4.2.4}$$

with $j_4 \neq j_1$, and prove $c_{j_1,j_2}\big(\Lambda(E)\big) = c_{j_4,j_3}\big(\Lambda(E)\big)$ By what we have just proved, for given j_1, j_2, j_4 it suffices to prove it for a specific j_3 satisfying (4.2.4), as we have proved that the coefficient $c_{j_4,j_3}\big(\Lambda(E)\big)$ is independent of such j_3. We can consider the symmetry $S_{P_{j_1},P_{j_4}}$ and put $P_{j_3} = S_{P_{j_1},P_{j_4}}(P_{j_2})$, and repeat the same symmetry argument as before, so that in fcat $c_{j_1,j_2}\big(\Lambda(E)\big) = c_{j_4,j_3}\big(\Lambda(E)\big)$. $\blacksquare$

Remark 4.2.6. It follows from Remark 4.2.5 and Remark 4.2.1 that the quantity

$$\sum_{j \neq j_1} c_{j,j_1}\Big(\Lambda(E)\Big)$$

is independent of j_1. $\blacksquare$

4.3 Probabilistic Interpretation of Λ.

Given $E \in \widetilde{\mathcal{D}}$, we put $c(Q, Q) = 0$,

$$c(Q_1, Q_2) = \begin{cases} c_{j_1,j_2} & \text{if} \, \exists \, i = 1, .., k, j_1, j_2 = 1, .., N, j_1 \neq j_2 : Q_1 = \psi_i(P_{j_1}), Q_2 = \psi_i(P_{j_2}) \\ 0 & \text{otherwise} \end{cases}$$

for every $Q_1, Q_2 \in V^{(1)}$, where, of course, c_{j_1,j_2} are the coefficients of E. Note that, in view of e), there exists at most one i satisfying the previous requirements, thus such a definition is correct. The number $c(Q_1, Q_2)$ can be interpreted as the *conductivity* from Q_1 to Q_2.

From now on we will assume that the coefficients of E are in $\mathcal{H}$,

in particular, ii) in the definition of $\mathcal{H}$ holds. The coefficient c_{j_1,j_2} can be interpreted as the probability that an object staying at P_{j_1} and moving through $V^{(0)}$, moves to P_{j_2} at the successive step. In fact, the sum of the probabilities of moving from P_{j_1} to the different points of $V^{(0)}$ is 1 by ii). We could guess that by this point of view, $c(Q_1, Q_2)$ represents the probability of moving in $V^{(1)}$ from Q_1 to Q_2. However, this is not precise, as simple examples show that, for $Q_1 \in V^{(1)}$, the sum $\sum\limits_{Q_2 \in V^{(1)}} c(Q_1, Q_2)$ is not necessarily 1. The reason is that Q_1 could belong to different 1-cells, thus the sum is 1 over every of the cells it belongs to. Thus, we need a modification of the previous notion, namely we put

$$\delta(Q_1, Q_2) := \frac{c(Q_1, Q_2)}{\sum\limits_{Q \in V^{(1)}} c(Q_1, Q)} .$$

We clearly have

$$\sum_{Q_2 \in V^{(1)}} \delta(Q_1, Q_2) = 1 \tag{4.3.1}$$

so that we can interpret $\delta(Q_1, Q_2)$ as the probability that a point staying at Q_1 and moving through $V^{(1)}$, moves to Q_2. We put $Q_1 \underset{E}{\sim} Q_2$ if $c(Q_1, Q_2) = 0$, that amounts to $\delta(Q_1, Q_2) = 0$, and in such a case we will say that Q_1 is E-close to Q_2. Note that if Q_1 and Q_2 do not lie in a common 1-cell, then they are not E-close. However, when Q_1 and Q_2 lie in a common cell, they are not necessarily E-close, as the coefficient relating them could be 0. If there exists $i = 1, ..., k$ such that $Q_1, Q_2 \in V_i$, we say that Q_1 and Q_2 are close and write $Q_1 \sim Q_2$. A trivial but important consequence of the previous discussion, which directly follows from (4.3.1) is that

$$\forall Q \in V^{(1)} \, \exists Q' \in V^{(1)} : \delta(Q, Q') > 0 . \tag{4.3.2}$$

We will now define the probability that a point staying at Q reaches a point P_{j_2} of $V^{(0)}$ not necessarily at the successive step, but following an arbitrary path. The function U we will now introduce is related to the notion of *Markov chain.* Given a path $X = (X_0, X_1, ..., X_n)$ we put

$$U(X)=\prod_{i=1}^{n}\delta(X_{i-1},X_i)$$

where by a *path* we mean a finite sequence $(X_i, i=0,...,n)$ of elements X_i of $V^{(1)}$, such that $X_{i-1}\sim X_i$ for every $i=1,...,n$. The last condition is not strictly necessary, but it is useful, and does not affect the definitions and results in the sequel as, if it does not hold then $U(X)=0$. If $n=0$, we put $U(X)=1$. In this situation, we say that n is the length of X and put $n=|X|$. We define $\mathcal{X}_n$ to be the set of the paths of length n and $\mathcal{X}:=\bigcup\limits_{n=1}^{+\infty}\mathcal{X}_n$. Moreover when $Q,Q'\in V^{(1)}$, $A\subseteq V^{(1)}$, we put

$$\mathcal{X}_n(Q)=\{X\in\mathcal{X}_n : X_0=Q\},\quad \mathcal{X}(Q)=\bigcup_{n=1}^{+\infty}\mathcal{X}_n(Q)\,,$$

$$\mathcal{X}_n(Q,Q')=\{X\in\mathcal{X}_n : X_0=Q, X_n=Q'\},\quad \mathcal{X}(Q,Q')=\bigcup_{n=1}^{+\infty}\mathcal{X}_n(Q,Q')\,,$$

$$\mathcal{X}_n(Q;A)=\{X\in\mathcal{X}_n : X_0=Q, X_i\notin A\ \ \forall i<n\},\quad \mathcal{X}(Q;A)=\bigcup_{n=1}^{+\infty}\mathcal{X}_n(Q,Q')\,,$$

$$\mathcal{X}_n(Q,Q';A)=\{X\in\mathcal{X}_n : X_0=Q, X_n=Q', X_i\notin A\ \ \forall i<n\}\,,$$

$$\mathcal{X}(Q,Q';A)=\bigcup_{n=1}^{+\infty}\mathcal{X}_n(Q,Q';A)\,.$$

Moreover, for every path X we define $X_{(m)}:=(X_0,X_1,...,X_m)$ if $m\le|X|$.

Lemma 4.3.1. *We have* $\sum\limits_{X\in\mathcal{X}_n(Q)}U(X)=1$ *for every positive integer n and for every $Q\in V^{(1)}$.*

Proof. We proceed by induction on n. If $n=1$, every $X\in\mathcal{X}_n(Q)$ has the form $X=(Q,Q')$ with $Q'\in V^{(1)}$, thus we have

$$\sum_{X\in\mathcal{X}_n(Q)}U(X)=\sum_{Q'\in V^{(1)}}\delta(Q,Q')=1$$

by (4.3.1). We now prove that, if the Lemma holds for n, it holds for $n+1$ as well. If $X\in\mathcal{X}_{n+1}(Q)$, then $X=(X_0,X_1,...,X_{n+1})$, where $X_0=Q$, X_1 is an arbitrary element of $V^{(1)}$, and $Y:=(X_1,...,X_{n+1})$ is an arbitrary element of $\mathcal{X}_n(X_1)$.With this notation we have $U(X)=\delta(Q,X_1)\delta(X_1,X_2)\cdots\delta(X_n,X_{n+1})=\delta(Q,X_1)U(Y)$, thus

$$\sum_{X\in\mathcal{X}_{n+1}(Q)}U(X)=\sum_{X_1\in V^{(1)}}\sum_{Y\in\mathcal{X}_n(X_1)}\delta(Q,X_1)U(Y)=\sum_{X_1\in V^{(1)}}\delta(Q,X_1)\Big(\sum_{Y\in\mathcal{X}_n(X_1)}U(Y)\Big)$$

$$= \sum_{X_1 \in V^{(1)}} \delta(Q, X_1) \qquad \text{by the inductive hypothesis}$$

$= 1$ by (4.3.1). Thus, the inductive step is proved, and the proof is complete. ∎

Lemma 4.3.2. *Given $Q, Q' \in V^{(1)}$, there exists $X \in \mathcal{X}(Q, Q')$ such that $U(X) > 0$.*

Proof. It suffices to prove that there exists a path $X \in \mathcal{X}(Q, Q')$ such that X_{i-1} and X_i are E-close for every $i = 1, ..., |X|$, in other words that any two points of $V^{(1)}$ are E-connected, using notation of [P]. To see this, it suffices to use the proof of Lemma 3.8 in [P]. There, it is proved that Q and Q' are E-connected and the hypothesis $Q' \in V^{(0)}$, which was mentioned there, is in fact never used in the proof. ∎

Lemma 4.3.3. *There exists a positive integer M such that, for every $Q, Q' \in V^{(1)}$, we have* $\sum_{X \in \mathcal{X}_M(Q;\{Q'\})} U(X) < 1$.

Proof. For every $Q, Q' \in V^{(1)}$ there exists $Y \in \mathcal{X}(Q, Q')$ such that $U(Y) > 0$, by Lemma 4.3.2, and of course for every $M > |X|$ there exists $X \in \mathcal{X}_M(Q)$ such that $X_n = Q'$, and $U(X) > 0$, where $n = |X|$, as it suffices to take X such that $X_i = Y_i$ for $i \le n$, and for $i > n$ we choose, using (4.3.2), $X_i \in V^{(1)}$ inductively in such a way that $\delta(X_{i-1}, X_i) > 0$. The path X and its length $|X| =: n(Q, Q')$, a priori, depend on Q, Q'. We take $M > \max\{n(Q, Q') : Q, Q' \in V^{(1)}\}$. Then for every $Q, Q' \in V^{(1)}$, there exists $\bar{X} \in \mathcal{X}_M(Q)$ such that $\bar{X}_n = Q'$ for some $n < M$, and $U(\bar{X}) > 0$. Thus, $\bar{X} \in \mathcal{X}_M(Q) \setminus \mathcal{X}_M(Q;\{Q'\})$, and, in view of Lemma 4.3.1,

$$\sum_{X \in \mathcal{X}_M(Q;\{Q'\})} U(X) \le \Big(\sum_{X \in \mathcal{X}_n(Q)} U(X) \Big) - U(\bar{X}) = 1 - U(\bar{X}) < 1 . \qquad ∎$$

Let $\alpha_n := \max_{Q_1, Q_2 \in V^{(1)}, Q_1 \ne Q_2} \sum_{X \in \mathcal{X}_n(Q_1;\{Q_2\})} U(X)$. By Lemma 4.3.1 we have $\alpha_n \le 1$ for all n. We have

Lemma 4.3.4.
i) The sequence α_n is decreasing.
ii) $\alpha_{n+m} \le \alpha_n \alpha_m$ for every natural numbers n, m.

Proof. We prove ii). Let $n, m > 0$ and let $X \in \mathcal{X}_{n+m}(Q_1;\{Q_2\})$, $Y(X) = (X_0, X_1, ..., X_n)$, $Z(X) = (X_n, X_{n+1}, ..., X_{n+m})$. Then, $U(X) = U(Y(X))U(Z(X))$, and $Y(X) \in \mathcal{X}_n(Q_1;\{Q_2\})$, $Z(X) \in \mathcal{X}_m(Q(X);\{Q_2\})$. Hence

$$\sum_{X \in \mathcal{X}_{n+m}(Q_1;\{Q_2\})} U(X) = \sum_{X \in \mathcal{X}_n(Q_1;\{Q_2\})} U(Y(X))U(Z(X)) \le$$

$$\sum_{Y \in \mathcal{X}_n(Q_1;\{Q_2\}), Z \in \mathcal{X}_m(Y_n;\{Q_2\})} U(Y)U(Z) =$$

$$\sum_{Y \in \mathcal{X}_n(Q_1;\{Q_2\})} \Big(U(Y) \sum_{Z \in \mathcal{X}_m(Y_n;\{Q_2\})} U(z) \Big) \le$$

$$\sum_{Y\in\mathcal{X}_n(Q_1;\{Q_2\})}\Big(U(Y)\max_{Q_3\in V^{(1)}\setminus\{Q_2\}}\sum_{Z\in\mathcal{X}_m(Q_3;\{Q_2\})}U(Z)\Big)$$
$$\leq\sum_{Y\in\mathcal{X}_n(Q_1;\{Q_2\})}\alpha_m U(Y)\leq\alpha_n\alpha_m\,,$$

and ii) is proved. As $\alpha_1\leq 1$, we have $\alpha_{n+1}\leq\alpha_n\alpha_1\leq\alpha_n$ and i) is proved. ■

Fix now $P_{j_1},P_{j_2}\in V^{(0)}$ with $j_1\neq j_2$ and define $W(=W_{j_1,j_2}):V^{(1)}\mathbb{R}$ by

$$W(Q):=\sum_{n=0}^{+\infty}W_n(Q)\tag{4.3.3}$$

$$W_n(Q):=\sum_{X\in\mathcal{X}_n(Q,P_{j_2};V^{(1)}\setminus\{P_{j_1}\})}U(X)\qquad\forall\,Q\in(V^{(1)}\setminus V^{(0)})\cup\{P_{j_1}\}\,,\tag{4.3.4}$$

$$W_n(P_{j_2})=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n>0\end{cases},\quad W_n(P_j)=0\quad\forall j\neq j_1,j_2\,.\tag{4.3.5}$$

The meaning of such a function W is that W represents the probability for a point staying at Q and moving through $V^{(1)}$, that the first element of $V^{(0)}\setminus P_{j_1}$ it meets is P_{j_2}. By this point of view, formulas in (4.3.5) are natural. We have to check that the series in (4.3.3) is convergent. In fact, the sum in (4.3.3) is not bigger than

$$\sum_{n=0}^{+\infty}\alpha_n\,.\tag{4.3.6}$$

On the other hand, by Lemma 4.3.3 we have $\alpha_M:=\bar{l}<1$, and by Lemma 4.3.4 ii), we have $\alpha_{M^h}\leq\bar{l}^h$. Thus, by Lemma 4.3.4 i), we have $\alpha_n\leq 1$ if $n<M$, $\alpha_n\leq\bar{l}$ if $M\leq n<2M$, $\alpha_n\leq\bar{l}^2$ if $2M\leq n<3M$ and so on, therefore the sum in (4.3.6) is

$$\alpha_0+\cdots+\alpha_{M-1}+\alpha_M+\cdots+\alpha_{2M-1}+\alpha_{2M}+\cdots+\alpha_{3M-1}+\cdots$$
$$\leq M+M\bar{l}+M\bar{l}^2+M\bar{l}^3+.....<+\infty\,,$$

as we have a geometric series of ratio $\bar{l}<1$. Now note that, when $n=0$ we have no summand in (4.3.4), as $X=(X_0)$, but $X_0=Q$ and $X_0=X_n=P_{j_2}\neq Q$, as $P_{j_2}\in V^{(0)}$ and $P_{j_2}\neq P_{j_1}$, thus $P_{j_2}\notin(V^{(1)}\setminus V^{(0)})\cup\{P_{j_1}\}$. Therefore, $W_0(Q)=0$, and in the sum in (4.3.3) we can consider $n\geq 1$.

Lemma 4.3.5. *We have*

$$W(Q)=\sum_{Q'\in V^{(1)}}\delta(Q,Q')\,W(Q')\quad\forall\,Q\in(V^{(1)}\setminus V^{(0)})\cup\{P_{j_1}\}\,.$$

Proof. Let $Q\in(V^{(1)}\setminus V^{(0)})\cup\{P_{j_1}\}$. Every summand in (4.3.4) with $n>0$, is $U(X)$ where $X=(Q,X_1,...,X_n)$ and

$$U(X) = \delta(Q, X_1)U(X_1, ..., X_n),\ X_n = P_{j_2}\,. \tag{4.3.7}$$

It follows

$$W_n(Q) = \sum_{Q' \in V^{(1)}} \delta(Q, Q')\, W_{n-1}(Q') \tag{4.3.8}$$

for every $n > 0$. In order to prove (4.3.8), we split the sum in (4.3.4) into three sums, $\Sigma_1 + \Sigma_2 + \Sigma_3$, where Σ_1 is the sums of those summands in (4.3.4) with X such that $X_1 \in (V^{(1)} \setminus V^{(0)}) \cup \{P_{j_1}\}$, Σ_2 is the sums of those summands in (4.3.4) with X such that $X_1 = P_{j_2}$ and Σ_3 is the sums of those summands in (4.3.4) with X such that $X_1 = P_j$, $j \neq j_1, j_2$. By (4.3.7) and the definition of W_{n-1}, we have

$$\Sigma_1 = \sum_{Q' \in (V^{(1)} \setminus V^{(0)}) \cup \{P_{j_1}\}} \delta(Q, Q')\, W_{n-1}(Q')\,. \tag{4.3.9}$$

In the summands in Σ_2, we have $n = 1$, as by definition of

$$\mathcal{X}_n(Q, P_{j_2}; V^{(1)} \setminus \{P_{j_1}\})\,,$$

we have $X_i \notin V^{(1)} \setminus \{P_{j_1}\}$, thus, in particular, $X_i \neq P_{j_2}$ for all $i < n$, and on the other hand, $X_1 = P_{j_2}$. Therefore,

$$\Sigma_2 = \delta(Q, P_{j_2}) = \delta(Q, P_{j_2}) W_{n-1}(P_{j_2})\,. \tag{4.3.10}$$

There are no summands in Σ_3, as, on one hand, we have $n = 1$ for the same reason as before, so that $X_1 = X_n = P_{j_2}$, and on the other, $X_1 \neq P_{j_2}$ by the definition of Σ_3. Therefore

$$\Sigma_3 = 0 = \sum_{Q' \in V^{(0)} \setminus \{P_{j_1}, P_{j_2}\}} \delta(Q, Q')\, W_{n-1}(Q')\,. \tag{4.3.11}$$

By summing up (4.3.9), (4.3.10) and (4.3.11), we get (4.3.8), thus the Lemma. ∎

We note that

$$S_1(E)(v) = \sum_{i=1}^{k} E(v \circ \psi_i) = \sum_{i=1}^{k} \sum_{j_1 < j_2} c_{j_1, j_2} \Big(v\big(\psi_i(P_{j_1})\big) - v\big(\psi_i(P_{j_2})\big) \Big)^2 =$$

$$\Rightarrow S_1(E)(v) = \sum_{Q_1, Q_2 \in V^{(1)}} c(Q_1, Q_2) \big(v(Q_1) - v(Q_2) \big)^2 \tag{4.3.12}$$

by the definition of $c(Q_1, Q_2)$. We have

Theorem 4.3.6. *The function W_{j_1, j_2} is the unique function that minimizes $S_1(E)$ on the set*

$$\mathcal{L}'_{j_1,j_2} := \{v \in \mathbb{R}^{V^{(1)}} : v(P_{j_2}) = 1, v(P_j) = 0 \quad \forall\, j \neq j_1, j_2\}\,.$$

Proof. (Hint) In the same way as in the proof that $H_{1;E}(u)$ is the function in $\mathcal{L}(u)$ satisfying formula (3.5) in Lemma 3.9 of [P], in view of (4.3.12), we can prove that the unique function that minimizes $S_1(E)$ in $\mathcal{L}'_{j_1,j_2}$ is the function v satisfying

$$\sum_{Q\in V^{(1)}} c(\bar{Q}, Q)\big(v(\bar{Q}) - v(Q)\big) = 0 \quad \forall\, \bar{Q} \in (V^{(1)} \setminus V^{(0)}) \cup \{P_{j_1}\}\,.$$

But, for every $\bar{Q} \in (V^{(1)} \setminus V^{(0)}) \cup \{P_{j_1}\}$, the previous formula amounts to

$$v(\bar{Q}) = \sum_{Q\in V^{(1)}} \frac{c(\bar{Q}, Q)}{\sum\limits_{Q'\in V^{(1)}} c(\bar{Q}, Q')} v(Q) = \sum_{Q\in V^{(1)}} \delta(\bar{Q}, Q) v(Q)\,. \tag{4.3.13}$$

Now, by Lemma 4.3.5, W_{j_1,j_2} satisfies (4.3.13), and by its definition $W_{j_1,j_2} \in \mathcal{L}'_{j_1,j_2}$, hence $v = W_{j_1,j_2}$. ▪

Corollary 4.3.7. *The restriction $\tilde{W}$ of W_{j_1,j_2} to $V^{(0)}$, minimizes $\Lambda(E)$ on the set*

$$\mathcal{L}''_{j_1,j_2} := \{u \in \mathbb{R}^{V^{(0)}} : u(P_{j_2}) = 1, u(P_j) = 0 \quad \forall\, j \neq j_1, j_2\}\,.$$

Proof. Let $v \in \mathcal{L}(\tilde{W})$. Then, as $v = \tilde{W} = W_{j_1,j_2}$ on $V^{(0)}$, and $W_{j_1,j_2} \in \mathcal{L}'_{j_1,j_2}$, by the definition of $\mathcal{L}'_{j_1,j_2}$, we have $v \in \mathcal{L}'_{j_1,j_2}$. Thus, by Theorem 4.3.6, $S_1(E)(v) \geq S_1(E)(W_{j_1,j_2})$. Then, by definition of Λ, we have

$$S_1(E)(W_{j_1,j_2}) = \Lambda(E)(\tilde{W})\,. \tag{4.3.14}$$

Given $u \in \mathcal{L}''_{j_1,j_2}$, we have $H_{1;E}(u) \in \mathcal{L}'_{j_1,j_2}$, hence

$$\begin{aligned} \Lambda(E)(u) = S_1(E)(H_{1;E}(u)) &\geq S_1(E)(W_{j_1,j_2}) && \text{by Theorem 4.3.6} \\ &= \Lambda(E)(\tilde{W}) \quad ▪ && \text{by (4.3.14)} \end{aligned}$$

We put $E' = \Lambda(E)$, and we want to evaluate $\tilde{W}$ at all points of $V^{(0)}$. As the values of $\tilde{W}(P_j) = W_{j_1,j_2}(P_j)$, by the definition of W_{j_1,j_2} are prescribed for all $j \neq j_1$, we have only to evaluate $W_{j_1,j_2}(P_{j_1})$. By proceeding as in Theorem 4.3.6, we have that $\tilde{W}$ satisfies

$$\sum_{j\neq j_1} c_{j_1,j}(E')\big(\tilde{W}(P_{j_1}) - \tilde{W}(P_j)\big) = 0$$

that amounts to

$$W_{j_1,j_2}(P_{j_1}) = \tilde{W}(P_{j_1}) = \sum_{j\neq j_1} \frac{c_{j_1,j}(E')}{\sum\limits_{j'\neq j_1} c_{j_1,j'}(E')} \tilde{W}(P_j) =$$

$$\sum_{j\neq j_1} \frac{c_{j_1,j}(E')}{\sum\limits_{j'\neq j_1} c_{j_1,j'}(E')} W_{j_1,j_2}(P_j) = \frac{c_{j_1,j_2}(E')}{\sum\limits_{j'\neq j_1} c_{j_1,j'}(E')}$$

recalling that $W_{j_1,j_2}(P_{j_2}) = 1$ and $W_{j_1,j_2}(P_j) = 0$ if $j \neq j_1, j_2$. Now, suppose

$$W_{j_1,j_2}(P_{j_1}) = c_{j_1,j_2} \quad \forall\, j_1, j_2\,. \tag{4.3.15}$$

Then we have $c_{j_1,j_2}\big(\Lambda(E)\big) = \rho c_{j_1,j_2}(E)$, where $\rho = \sum\limits_{j'\neq j_1} c_{j_1,j'}(E')$, which, by Remark 4.2.6, is independent of j, hence E is an eigenform. Thus, the problem of the existence of an eigenform is reduced to the problem of the existence of a fixed point of the map

$$(c_{j_1,j_2}) \mapsto \big(W_{j_1,j_2}(P_{j_1})\big) \tag{4.3.16}$$

The nontrivial point consists in proving that such a map in fact sends $\mathcal{H}$ into itself, and this will be the aim of next section. We are now giving a probabilistic interpretation of the fixed points of the map in (4.3.16). Recalling the meaning of $W_{j_1,j_2}(Q)$, (c_{j_1,j_2}) is a fixed point of the map in (4.3.16) if the probability for a point staying at P_{j_1} that the first point of $V^{(0)}$ it reaches is P_{j_2} is the same if the point moves through $V^{(0)}$ and if the points moves through $V^{(1)}$, and it is not difficult to prove that this implies that the probability is the same if the point moves through $V^{n)}$, for every natural n, in other words the probability of a point staying at P_{j_1} that the first point of $V^{(0)}$ it reaches is P_{j_2} is independent of the scale. Those considerations have suggested the notion of *Brownian motion* on the fractal. However, we will not insist on this point.

4.4 Lindstrøm Theorem.

In this Section, we will prove the Lindstrøm Theorem, that is that, on any nested fractal there exists an eigenform. By the previous considerations, it suffices to prove that given $(c_{j_1,j_2}) \in \mathcal{H}$, then $\big(W_{j_1,j_2}(P_{j_1})\big) \in \mathcal{H}$. The only nontrivial point is iii) of the definition of $\mathcal{H}$, i) being a consequence of iii), and ii) being a consequence of the previously proved formula

$$W_{j_1,j_2}(P_{j_1}) = \frac{c_{j_1,j_2}(E')}{\sum\limits_{j'\neq j_1} c_{j_1,j'}(E')}, \quad E' := \Lambda(E)\,, \tag{4.4.1}$$

Theorem 4.4.1. *If* $(c_{j_1,j_2}) \in \mathcal{H}$, *then* $\big(W_{j_1,j_2}(P_{j_1})\big) \in \mathcal{H}$.

Proof. As noted above, it suffices to prove that $\big(W_{j_1,j_2}(P_{j_1})\big)$ satisfies iii) in the definition of $\mathcal{H}$. By (4.4.1), Remarks 4.2.5 and 4.2.6, W_{j_1,j_2} only depends on $||P_{j_1} - P_{j_2}||$, hence it suffices to prove that if

$$||P_{j_1} - P_{j_3}|| > ||P_{j_1} - P_{j_2}|| > 0 \tag{4.4.2}$$

then

$$W_{j_1,j_2}(P_{j_1}) \geq W_{j_1,j_3}(P_{j_1}) \,. \tag{4.4.3}$$

Let $H = H_{P_{j_2},P_{j_3}}$ $S = S_{P_{j_2},P_{j_3}}$, let A be the closed half-space bounded by H containing P_{j_2} and B be the closed the half-space bounded by H containing P_{j_3}. By (4.4.2), $P_{j_1} \in A$. Note that

$$Q_1 \sim Q_2 \Rightarrow S(Q_1) \sim S(Q_2)$$

as, if $Q_1 \sim Q_2$ then $Q_1, Q_2 \in V_i$ for some $i = 1, ..., k$, and $S(Q_1), S(Q_2) \in S(V_i)$, but d) of the definition of nested fractal implies that $S(V_i)$ is a 1-cell, so that $S(Q_1) \sim S(Q_2)$, as claimed. Let T be defined by

$$T(x) = \begin{cases} x & \text{if } x \in A \\ S(x) & \text{if } x \in B \end{cases}$$

and let $\mathcal{Y}_2 = \mathcal{X}(P_{j_1}, P_{j_2}; V^{(0)} \setminus \{P_{j_1}\})$, $\mathcal{Y}_3 = \mathcal{X}(P_{j_1}, P_{j_3}; V^{(0)} \setminus \{P_{j_1}\})$. By the definition of W_{j_1,j_2}, then (4.4.3) amounts to

$$\sum_{X \in \mathcal{Y}_2} U(X) \geq \sum_{X \in \mathcal{Y}_3} U(X) \tag{4.4.4}$$

Now,

$$Q_1 \sim Q_2 \Rightarrow T(Q_1) \sim T(Q_2) \tag{4.4.5}$$

In fact, if $Q_1, Q_2 \in A$, then (4.4.4) is trivial. If $Q_1, Q_2 \in B$ then $T(Q_1) = S(Q_1) \sim S(Q_2) = T(Q_2)$. If $Q_1 \notin B$, $Q_2 \notin A$, then $T(Q_1) = Q_1$, $T(Q_2) = S(Q_2)$, and the 1-cell V_i containing Q_1 and Q_2, contains then points on both sides on H, thus by the hypothesis, S maps V_i into itself, hence $T(Q_2) = S(Q_2) \in V_i$, and $T(Q_1) = Q_1 \in V_i$ by hypothesis, so that $T(Q_1) \sim T(Q_2)$, and (4.4.5) is proved. Moreover,

$$Q_1 \sim Q_2 \Rightarrow \delta(T(Q_1), T(Q_2)) \geq \delta(Q_1, Q_2) \tag{4.4.6}$$

In fact, (4.4.6) is trivial if $Q_1, Q_2 \in A$, but it is also clear if $Q_1, Q_2 \in B$. In fact, $c(Q_1, Q) = c\big(S(Q_1), S(Q)\big)$ for every $Q \sim Q_1$, as S preserves the distance and the coefficients $c_{j,j'}$ only depend on the distance. If finally, $Q_1 \notin B$, $Q_2 \notin A$, then, $Q_1, Q_2, S(Q_2)$, as previously seen, lie in a common 1-cell, and by the definition of S, $||S(Q_2) - Q_1|| \leq ||Q_2 - Q_1||$, so that $c\big(T(Q_1), T(Q_2)\big) = c\big(Q_1, S(Q_2)\big) \geq c(Q_1, Q_2)$, and (4.4.6) is proved. Given $X \in \mathcal{Y}_2 \cup \mathcal{Y}_3$, $X = (X_0, X_1, ..., X_n)$, we put $T(X) = (T(X_0), T(X_1), ..., T(X_n))$. Note that $T(X) \in \mathcal{Y}_2$ for every $X \in \mathcal{Y}_2 \cup \mathcal{Y}_3$. Now, for any $Y \in \mathcal{Y}_2$ such that $Y_i \in A$ for each i, let $n = |Y|$, and put

$$\mathcal{Y}_2(Y) = \{X \in \mathcal{Y}_2 : T(X) = Y\}, \quad \mathcal{Y}_3(Y) = \{X \in \mathcal{Y}_3 : T(X) = Y\} \,.$$

Note that the hypothesis on Y implies $T(Y_i) = Y_i$ for each i. Clearly, in order to prove (4.4.4), hence the Theorem, it suffices to prove

$$\sum_{X \in \mathcal{Y}_2(Y)} U(X) \geq \sum_{X \in \mathcal{Y}_3(Y)} U(X) \tag{4.4.7}$$

Note that every $X \in \mathcal{Y}_2(Y) \cup \mathcal{Y}_3(Y)$ has the form $X_i = S^{\alpha_{i,X}}(Y_i)$, where $\alpha_{i,X} = 0, 1$. We use the convention that, if $X_i \in A \cap B$, so that $S(X) = X$, and $\alpha_{i,X}$ thus can be arbitrary, then $\alpha_{i,X} = \alpha_{i-1,X}$. Note that, if $X \in \mathcal{Y}_2(Y) \cup \mathcal{Y}_3(Y)$, then $X_0 = P_{j_1} = Y_0$, thus $\alpha_{0,X} = 0$. On the other hand, If $X \in \mathcal{Y}_2(Y)$, then, $X_n = P_{j_2} = Y_n$, thus $\alpha_{n,X} = 0$, while if $X \in \mathcal{Y}_3(Y)$, then, $X_n = P_{j_3} = S(Y_n)$, thus $\alpha_{n,X} = 1$. Thus, putting

$$J_X = \{i : \alpha_i(X) \neq \alpha_{i-1}(X)\},$$

then $\#(J_X)$ is even if $X \in \mathcal{Y}_2(Y)$, and odd if $X \in \mathcal{Y}_3(Y)$. However J_X is not an arbitrary subset of $\{1, ..., n\}$, Namely it is a subset of

$$J := \big\{i = 1, ..., n : Y_i \notin A \cap B, S(Y_i) \sim Y_{i-1}\big\}$$

and, conversely, it is easy to see that, for every subset of $\tilde{J}$ of J, there exists exactly one $X \in \mathcal{Y}_2(Y) \cup \mathcal{Y}_3(Y)$ such that $J_X = \tilde{J}$, and moreover $X \in \mathcal{Y}_2(Y)$ if $\tilde{J}$ is even, and $X \in \mathcal{Y}_3(Y)$ if $\tilde{J}$ is odd. Moreover, if $X \in \mathcal{Y}_2(Y) \cup \mathcal{Y}_3(Y)$, we have

$$U(X) = U(Y) \prod_{i \in J(X)} \eta_i, \quad \eta_i = \frac{\delta(Y_{i-1}, S(Y_i))}{\delta(Y_{i-1}, Y_i)} \leq 1$$

We have $\eta_i \leq 1$ as $T(S(Y_i)) = Y_i$, thus by (4.4.6), $\delta(Y_{i-1}, Y_i) = \delta\big(T(Y_{i-1}), T(S(Y_i))\big) \geq \delta\big(Y_{i-1}, S(Y_i)\big)$ by (4.4.6). Thus,

$$\begin{aligned}
\sum_{X \in \mathcal{Y}_2} U(X) - \sum_{X \in \mathcal{Y}_3} U(X) &= U(Y)\Big(\sum_{\tilde{J} \subseteq J,\ \#(\tilde{J})\,\text{even}} \prod_{i \in \tilde{J}} \eta_i - \sum_{\tilde{J} \subseteq J,\ \#(\tilde{J})\,\text{odd}} \prod_{i \in \tilde{J}} \eta_i\Big) \\
&= U(Y)\Big(1 - \sum_{i \in J} \eta_i + \sum_{i_1, i_2 \in J} \eta_{i_1}\eta_{i_2} - \sum_{i_1, i_2, i_3 \in J} \eta_{i_1}\eta_{i_2}\eta_{i_3} + \cdots\Big) \\
&= U(Y) \prod_{i \in J} \big(1 - \eta_i\big) \geq 0
\end{aligned}$$

where in the sums we mean that i_1, i_2, i_3 are mutually different, and (4.4.7) holds, thus the Theorem is proved. To be precise, the previous argument does not hold when $U(Y) = 0$, as η_i could contain a 0-denominator, but in such a case, the same argument shows that every $U(X)$ is 0, so (4.4.7) holds in this case too. ∎

Corollary 4.4.2. *There exists an eigenform on every nested fractal.*

Proof. By Theorem 4.4.1, the map $(c_{j_1,j_2}) \mapsto \big(W_{j_1,j_2}(P_{j_1})\big)$ maps $\mathcal{H}$ into itself. Such a map is continuous, by Remark 4.2.4 and (4.4.1). As previously observed, the set $\mathcal{H}$ is compact, convex and nonempty, therefore it has a fixed point. By the considerations in the end of Section 4.3, Λ has an eigenform. ∎