3. Geometry of Fractals

3.1 Hausdorff Measures

Recall that an *outer measure (or exterior measure)* on a set X is a function μ defined on $\mathcal{P}(X)$ (the set of the subsets of X) with values in $[0, +\infty]$, satisfying the following properties:
i) $\mu(\emptyset) = 0$.
ii) If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
iii) For every sequence A_i of subsets of X, we have $\mu\Big(\bigcup\limits_{i=1}^{+\infty} A_i\Big) \leq \sum\limits_{i=1}^{+\infty} \mu(A_i)$.
There are two main differences between the notion of outer measure, and the notion of measure. One is that an outer measure is a function defined on all $\mathcal{P}(X)$, while a measure is defined on a σ-algebra. The other difference is that an outer measure is σ-subadditive (property iii)), but not necessarily additive. If μ is an outer measure on a set X, then a subset E of X is called *mesasurable* (with respect to μ) if it satisfies

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E) \quad \forall\, A \subseteq X\,. \tag{3.1.1}$$

As well known from the general measure theory, the set $\mathcal{M}$ of the measurable sets is a σ-algebra, and the restriction of μ to $\mathcal{M}$ is a measure.

We are now going to define the Hausdorff measure. For every subset A of a metric space X and $\mathcal{E} \subseteq \mathcal{P}(X)$, for every $\alpha \geq 0$, $\delta > 0$, we put

$$H'_{\alpha,\delta;\mathcal{E}}(A) = \inf \Big\{ h_\alpha((S_i)) | (S_i) \in \mathcal{S}_{\delta,A,\mathcal{E}} \Big\}$$

$$\mathcal{S}_{\delta,A,\mathcal{E}} := \Big\{ (S_i) : S_i \in \mathcal{E}, \bigcup_{i=1}^{+\infty} S_i \supseteq A, \mathrm{diam}(S_i) \leq \delta \Big\}, \quad h_\alpha((S_i)) := \sum_{i=1}^{+\infty} \big(\mathrm{diam}(S_i)\big)^\alpha,$$

$$H'_{\alpha;\mathcal{E}}(A) = \lim_{\delta \to 0^+} H'_{\alpha,\delta;\mathcal{E}}(A)$$

with the usual convention $\inf(\emptyset) = +\infty$, where we intend in $((S_i))$ that $i = 1, 2, 3, \ldots$. When $\mathcal{E} = \mathcal{P}(X)$, we could omit $\mathcal{E}$ and simply write $H'_{\alpha,\delta}(A)$, $H'_\alpha(A)$, $\mathcal{S}_{\delta,A}$. Note that the limit in the definition of $H'_{\alpha;\mathcal{E}}(A)$ does exist as we can easily verify that $H'_{\alpha,\delta;\mathcal{E}}(A)$ is decreasing with respect to δ. The α dimensional (or simply α) Hausdorff measure H_α is given by

$$H_\alpha = K_\alpha H'_\alpha$$

where K_α is a suitable positive constant. Such a constant is chosen so that the n-dimensional Hausdorff measure in $\mathbb{R}^n$ amounts to Lebesgue measure. Note that, at this point, we cannot state that such a positive constant really exists, as we could also guess that H'_n is identically 0 on $\mathbb{R}^n$. However, as we will see in Section 3.3, H'_n is in fact a positive multiple of Lebesgue measure so that such a constant K_n really exists. The value of the constant for positive, not integer α, is not important for our considerations. Note that in the definition of H' we take the infimum with countable coverings of A. If we would take the infimum with finite coverings, every unbounded set A would have measure $+\infty$, as it cannot be covered by finitely many sets of diameter not greater than δ. Note however, that the finite coverings are included in $\mathcal{S}_{\delta,A,\mathcal{E}}$ as it suffices to consider (S_i) with $S_i = \emptyset$ for $i > \bar{n}$. It is a standard fact from general measure theory that in fact $H'_{\alpha,\delta;\mathcal{E}}$, thus H'_α and H_α, are outer measures. In the following, we will often give results about H'_α, the corresponding ones for H_α being immediate consequences. Note that H'_0 is the measure counting the points, where we intend that $0^0 = 1$, with the exception that $(\operatorname{diam}(\emptyset))^0 = 0$.

Remark 3.1.1. If

$$\mathcal{E}' \subseteq \mathcal{E} \tag{3.1.2}$$

then trivially,

$$H'_{\alpha,\delta;\mathcal{E}}(A) \leq H'_{\alpha,\delta;\mathcal{E}'}(A) \Rightarrow H'_{\alpha;\mathcal{E}}(A) \leq H'_{\alpha;\mathcal{E}'}(A)\,. \tag{3.1.3}$$

If moreover,

$$\forall E \in \mathcal{E}\ \exists E' \in \mathcal{E}' : E' \supseteq E, \operatorname{diam}(E') = \operatorname{diam}(E) \tag{3.1.4}$$

then $H'_{\alpha,\delta;\mathcal{E}}(A) = H'_{\alpha,\delta;\mathcal{E}'}(A)$, therefore $H'_{\alpha;\mathcal{E}}(A) = H'_{\alpha;\mathcal{E}'}(A)$. In fact, put $D(E)$ to be the set E' given by (3.1.4), then for every $(S_i) \in \mathcal{S}_{\delta,A,\mathcal{E}}$, then $(D(S_i)) \in \mathcal{S}_{\delta,A,\mathcal{E}'}$ and $h_\alpha(D(S_i)) \leq h_\alpha((S_i))$. As an important particular case, defining $\mathcal{F}$ to be the set of the closed sets in X and putting $\mathcal{E} = \mathcal{P}(X)$, $\mathcal{E}' = \mathcal{F}$, we have

$$H'_{\alpha,\delta}(A) = H'_{\alpha,\delta;\mathcal{F}}(A) \tag{3.1.5}$$

that is, in the definition of $H'_{\alpha,\delta}$ we can assume that the sets are closed. It suffices in fact, to put $E' = \overline{E}$ in (3.1.4). If (3.1.4) is replaced by the more general

$$\forall \eta > 0\ \forall E \in \mathcal{E}\ \exists E' \in \mathcal{E}' : E' \supseteq E, \operatorname{diam}(E') \leq \operatorname{diam}(E) + \eta\,, \tag{3.1.6}$$

then

$$H'_{\alpha;\mathcal{E}}(A) = H'_{\alpha;\mathcal{E}'}(A)\,. \tag{3.1.7}$$

To see this, in view of (3.1.3) it suffices to prove that $\geq$ holds in (3.1.7) and we can clearly assume $H'_{\alpha;\mathcal{E}}(A) < +\infty$. Let $\varepsilon > 0$ and let $\bar{\delta} > 0$ be such that if $0 < \delta < \bar{\delta}$, then

$$H'_{\alpha,\delta;\mathcal{E}}(A) < H'_{\alpha;\mathcal{E}}(A) + \varepsilon\,. \tag{3.1.8}$$

Put $D(\eta, E)$ to be the set E' given by (3.1.6). For every $A \subseteq X$ and $(S_i) \in \mathcal{S}_{\delta,A,\mathcal{E}}$, by the continuity of the map $t \mapsto t^\alpha$, we have $(\operatorname{diam}(S_i) + \eta_i)^\alpha < (\operatorname{diam}(S_i))^\alpha + \frac{\varepsilon}{2^i}$ for sufficiently small, positive η_i. Let $S'_i = D(\eta_i, S_i)$, so that $(\operatorname{diam}(S'_i))^\alpha < (\operatorname{diam}(S_i))^\alpha + \frac{\varepsilon}{2^i}$. Therefore, by the definition of h_α we have

$$h_\alpha((S'_i)) \le h_\alpha((S_i)) + \sum_{i=1}^{+\infty} \frac{\varepsilon}{2^i} = h_\alpha((S_i)) + \varepsilon \tag{3.1.9}$$

Now, for every δ such that $0 < \delta < \bar{\delta}$, if $(S_i) \in \mathcal{S}_{\frac{\delta}{2},A,\mathcal{E}}$, and $\eta_i < \frac{\delta}{2}$, then $(S'_i) \in \mathcal{S}_{\delta,A,\mathcal{E}'}$, thus, $H'_{\alpha,\delta;\mathcal{E}'}(A) \le h_\alpha((S'_i)) \le h_\alpha((S_i)) + \varepsilon$. As this holds for every $(S_i) \in \mathcal{S}_{\frac{\delta}{2},A,\mathcal{E}}$, in view of (3.1.8), we have

$$H'_{\alpha,\delta;\mathcal{E}'}(A) \le H'_{\alpha,\frac{\delta}{2};\mathcal{E}}(A) + \varepsilon \le H'_{\alpha;\mathcal{E}}(A) + 2\varepsilon\,.$$

Hence, $H'_{\alpha;\mathcal{E}'}(A) \le H'_{\alpha;\mathcal{E}}(A) + 2\varepsilon$. By the arbitrarity of $\varepsilon > 0$, $\ge$ holds in (3.1.7). As an important particular case, defining $\mathcal{O}$ to be the set of the open sets in X, and putting $\mathcal{E} = \mathcal{P}(X)$, $\mathcal{E}' = \mathcal{O}$, we have

$$H'_{\alpha,\delta}(A) = H'_{\alpha,\delta;\mathcal{O}}(A) \tag{3.1.10}$$

that is, in the definition of $H'_{\alpha,\delta}$ we can assume that the sets are open. In fact, for every subset E of X, the set $E_{\frac{\eta}{2}} := \{x \in X : d(x, E) < \frac{\eta}{2}\}$ is open and contains E (cf. $K_{\frac{1}{h}}$ in Lemma 1.8.1), and moreover, $\mathrm{diam}\big(E_{\frac{\eta}{2}}\big) \le \mathrm{diam}(E) + \eta$, as, given two points $x, x' \in E_{\frac{\eta}{2}}$, then there exist $y, y' \in E$ such that $d(x, y) < \frac{\eta}{2}$, $d(x', y') < \frac{\eta}{2}$, thus

$$d(x, x') \le d(x, y) + d(y, y') + d(y', x') \le \frac{\eta}{2} + \mathrm{diam}(E) + \frac{\eta}{2} = \mathrm{diam}(E) + \eta\,.$$

Note that, if A is compact, by the very definition of compactness, we can consider in the definition of $H'_{\alpha,\mathcal{E}}$ only *finite* open coverings. ▪

Remark 3.1.2. By the essentially same argument we see that in the definition of $H'_{\alpha;\mathcal{E}}(A)$ we can require that the sets S_i are subsets of A. In fact, $h_\alpha((S_i \cap A)) \le h_\alpha((S_i))$. This means that the definition of $H'_\alpha(A)$ can be given considering A as a metric space, and not as a metric subspace of X. As a consequence, the definition of $H'_\alpha(A)$ does not depend on the metric space X. ▪

We now want to prove that the Borel sets are measurable with respect to the outer measure H'_α. We define $dist(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$. This notion, clearly, is very different from that of Haurdorff distance. In particular, $dist(A, B) = 0$ if $A \cap B \ne \emptyset$. Note also that if A and B are nonempty disjoint compact sets, then $dist(A, B) > 0$.

Lemma 3.1.3. *If A and B are nonempty subsets of X, and $dist(A, B) > 0$, then*

$$H_\alpha(A \cup B) = H_\alpha(A) + H_\alpha(B)\,.$$

Proof. We will prove the analogous result for H'_α, which is clearly equivalent. Let $\delta \in \left]0, \dfrac{dist(A, B)}{2}\right[$. Then, if

$$\Big(U \cap A \neq \emptyset, V \cap B \neq \emptyset, \mathrm{diam} U < \delta, \mathrm{diam} V < \delta\Big) \Rightarrow U \cap V = \emptyset\,. \tag{3.1.11}$$

In fact, in the opposite case, there would exist $\bar{x} \in U \cap V$, thus for every $x_1, x_2 \in U \cup V$, $d(x_1, x_2) \leq d(\bar{x}, x_1) + d(\bar{x}, x_2) \leq \delta + \delta$. Hence, if $x_1 \in U \cap A$, $x_2 \in V \cap B$, then $d(x_1, x_2) \leq 2\delta < dist(A, B)$, a contradiction. Let $(S_i) \in \mathcal{S}_{\delta, A\cup B}$, let

$$J := \{i = 1, 2, 3, ... : S_i \cap (A \cup B) \neq \emptyset\}\,,$$

$J_1 := \{i = 1, 2, 3, ... : S_i \cap A \neq \emptyset\}$, $J_2 := \{i = 1, 2, 3, ... : S_i \cap B \neq \emptyset\}$. Then, by (3.1.11), J is the union of the disjoint sets J_1 and J_2, thus

$$h_\alpha((S_i)) \geq \sum_{i\in J} (\mathrm{diam} S_i)^\alpha = \sum_{i \in J_1} (\mathrm{diam} S_i)^\alpha + \sum_{i\in J_2} (\mathrm{diam} S_i)^\alpha \geq H'_{\alpha,\delta}(A) + H'_{\alpha,\delta}(B)$$

thus, as (S_i) is an arbitrary element of $\mathcal{S}_{\delta, A\cup B}$, we have $H'_{\alpha,\delta}(A \cup B) \geq H'_{\alpha,\delta}(A) + H'_{\alpha,\delta}(B)$ and taking the limit, as $\delta \to 0^+$,

$$H'_\alpha(A \cup B) \geq H'_\alpha(A) + H'_\alpha(B)$$

and, as the opposite inequality is valid independently of A and B by the definition of outer measure, we have proved the Lemma. ▪

Theorem 3.1.4. *Every Borel set is measurable with respect to H_α.*

Proof. Since the set of the measurable sets is a σ-algebra, by the definition of Borel sets (the elements of the smallest σ-algebra containing the open sets), it suffices to prove that, for every nonempty open set $E \neq X$, then E is measurable, that is (3.1.1) holds. In (3.1.1), we have only to prove $\geq$, $\leq$ holding independently of A and E. Moreover, we can assume $H_\alpha(A) < +\infty$, the opposite case being trivial. Let

$$A_1 := \Big\{x \in A : d(x, X \setminus E) \geq 1\Big\}, \quad A_h := \Big\{x \in A : \frac{1}{h} \leq d(x; X \setminus E) < \frac{1}{h-1}\Big\}, h \geq 2\,.$$

It easily follows:

$$A \cap E = \bigcup_{i=1}^{+\infty} A_i\,. \tag{3.1.12}$$

$$H_\alpha\Big(\bigcup_{i\in J} A_i\Big) \leq \sum_{i\in J} H_\alpha(A_i) \quad \forall\, J \subseteq \{1, 2, 3, ...\}\,, \tag{3.1.13}$$

$$H_\alpha(A) \geq H_\alpha\Big(\bigcup_{i\in J} A_i\Big) \geq \sum_{i\in J} H_\alpha(A_i) \quad \forall\, J = J_1 \text{ or } J = J_2\,, \tag{3.1.14}$$

where $J_1 = \{1, 3, 5, 7, ...\}$, $J_2 = \{2, 4, 6, 8, ...\}$. To prove (3.1.12), note that every $x \in A \cap E$ has positive distance d from the closed set $X \setminus E$, thus either $d \geq 1$ or $\frac{1}{h} \leq d < \frac{1}{h-1}$ for some $h \geq 2$. Formula (3.1.13) is obvious as H_α is an outer measure. The first inequality in (3.1.14) immediately follows from (3.1.12). In order to prove the second inequality in (3.1.14), observe that for $j = 1, 2, 3, ...$ we have

$$dist\Big(A_{j+2}, \bigcup_{i \leq j} A_i\Big) \geq \frac{1}{j(j+1)} \tag{3.1.15}$$

Indeed, if $x \in \bigcup_{i \leq j} A_i$ and $y \in A_{j+2}$, then $d(x, X \setminus E) \geq \frac{1}{j}$ and $d(y, X \setminus E) \leq \frac{1}{j+1}$. As $d(x, X \setminus E) \leq d(x, y) + d(y, X \setminus E)$, it follows $d(x, y) \geq \frac{1}{j} - \frac{1}{j+1} = \frac{1}{j(j+1)}$ and (3.1.15) is proved. By (3.1.15) and Lemma 3.1.3, we have for every $h \geq 2$,

$$H_\alpha\Big((A_{2h}) \cup (\bigcup_{i<h} A_{2i})\Big) = H_\alpha(A_{2h}) + H_\alpha\Big(\bigcup_{i<h} A_{2i}\Big)$$

and, by induction, $H_\alpha\Big(\bigcup_{i \in J_2} A_i\Big) \geq H_\alpha\Big(\bigcup_{i \leq h} A_{2i}\Big) = \sum_{i=1}^{h} H_\alpha(A_{2i})$ and, taking the limit for $h \to +\infty$, we get (3.1.14) for $J = J_2$, and the argument for J_1 is essentially the same, so that (3.1.14) is proved. By (3.1.14), as we have assumed $H_\alpha(A) < +\infty$, we have $\sum_{i \in J, i > h} H_\alpha(A_i) \underset{h \to +\infty}{\longrightarrow} 0$ for $J = J_1, J_2$. Therefore,

$$H_\alpha\Big(\bigcup_{i=h+1}^{+\infty} A_i\Big) \leq \sum_{i=h+1}^{+\infty} H_\alpha(A_i) = \sum_{i \in J_1, i>h} H_\alpha(A_i) + \sum_{i \in J_2, i>h} H_\alpha(A_i) \underset{h \to +\infty}{\longrightarrow} 0\,. \tag{3.1.16}$$

Since $\mathrm{dist}(A \setminus E, \bigcup_{i=1}^{h} A_i) \geq \frac{1}{h}$ by the definition of A_i, by Lemma 3.1.3 we have

$$H_\alpha(A) \geq H_\alpha\big((A \setminus E) \cup (\bigcup_{i=1}^{h} A_i\big) \geq H_\alpha(A \setminus E) + H_\alpha\big(\bigcup_{i=1}^{h} A_i\big) \tag{3.1.17}$$

and, as $A \cap E = \Big(\bigcup_{i=1}^{h} A_i\Big) \cup \Big(\bigcup_{i=h+1}^{+\infty} A_i\Big)$ by (3.1.12), then

$$H_\alpha(A \cap E) \leq H_\alpha\Big(\bigcup_{i=1}^{h} A_i\Big) + H_\alpha\Big(\bigcup_{i=h+1}^{+\infty} A_i\Big)$$

and, by (3.1.17) we have

$$H_\alpha(A) \geq H_\alpha(A \setminus E) + H_\alpha(A \cap E) - H_\alpha\Big(\bigcup_{i=h+1}^{+\infty} A_i\Big)$$

so that, taking the limit for $h \to +\infty$, by (3.1.16), the inequality $\geq$ in (3.1.1) is proved. ∎

3.2 Hausdorff Dimension.

In this Section, we will define the Hausdorff dimension of a set in a metric space. This will turn out to be a nonnegative real number, not necessarily integer. We need a lemma.

Lemma 3.2.1 *If A is a subset of a metric space and $H_\alpha(A) < +\infty$, then $H_\beta(A) = 0$ for every $\beta > \alpha$.*
Proof. Let $(S_i) \in \mathcal{S}_{\delta,A}$. Then

$$h_\beta((S_i)) = \sum_{i=1}^{+\infty} \big(\mathrm{diam}(A_i)\big)^\beta = \sum_{i=1}^{+\infty} \big(\mathrm{diam}(A_i)\big)^\alpha \big(\mathrm{diam}(A_i)\big)^{\beta-\alpha}$$

$$\leq \delta^{\beta-\alpha} \sum_{i=1}^{+\infty} \big(\mathrm{diam}(A_i)\big)^\alpha = \delta^{\beta-\alpha} h_\alpha((S_i))$$

therefore, $H'_{\beta,\delta}(A) \leq \delta^{\beta-\alpha} H'_{\alpha,\delta}(A)$. As $\beta - \alpha > 0$, we have $\delta^{\beta-\alpha} \underset{\delta\to 0^+}{\longrightarrow} 0$, thus the Lemma easily follows taking the limit for $\delta \to 0^+$. ▪

We now put

$$M_I(A) = \{\alpha \geq 0 : H_\alpha(A) \in I\}, I \subseteq [0, +\infty]\,.$$

As a consequence of Lemma 3.2.1, if $\alpha \in M_{]0,+\infty[}(A)$ then every $\beta > \alpha$ lies in $M_{\{0\}}(A)$ and every $\beta < \alpha$ lies in $M_{\{+\infty\}}(A)$, so that $M_{]0,+\infty[}(A)$ is either empty or a singleton. Moreover, $M_{\{0\}}(A)$ is an upper half-line, that is, if it contains α, then it contains every $\beta > \alpha$, therefore it has the form $]\bar\alpha, +\infty[$ or $[\bar\alpha, +\infty[$ or is empty. Similarly, $M_{\{+\infty\}}(A)$ is a lower half-line, that is, if it contains α, then it contains every $\beta < \alpha$, therefore it has the form $[0, \bar\alpha[$ or $[0, \bar\alpha]$ or is empty, and, by Lemma 3.2.1 again, every element of $M_{\{0\}}(A)$ is greater than every element of $M_{\{+\infty\}}(A)$. It follows that $\inf M_{\{0\}}(A) = \sup M_{\{+\infty\}}(A) =: \dim_H(A)$, and $\dim_H(A)$ is called *Hausdorff dimension of A*. Here we use the standard convention $\inf(\emptyset) = +\infty$ and the less standard convention $\sup(\emptyset) = 0$. Clearly, if $0 < H_{\bar\alpha}(A) < +\infty$, then $\bar\alpha$ is the unique element of $M_{]0,+\infty[}(A)$ and amounts to $\dim_H(A)$. Note that the converse does not hold. In other words, if $\bar\alpha = \dim_H(A)$, we could have $H_{\bar\alpha}(A) = 0$ or $H_{\bar\alpha}(A) = +\infty$. To see this, as we know that the n-Husdorff measure amounts to Lebesgue measure, then $H_n([-k,k]^n) = (2k)^n$, then by Lemma 3.2.1 we have $H_\alpha([-k,k]^n) = 0$, thus $H_\alpha(\mathbb{R}^n) = 0$ for every $\alpha > n$, $H_\alpha([-k,k]^n) = +\infty$, thus $H_\alpha(\mathbb{R}^n) = +\infty$ for every $\alpha < n$, where we use the well known fact that the countable union of sets of 0-measure is a set of 0-measure and the countable union of sets of ∞-measure is a set of ∞-measure. Therefore, by definition $n = \dim_H(\mathbb{R}^n)$, but $H_n(\mathbb{R}^n) = +\infty$.

We will now sketch the definition of *topological dimension*. The idea of the definition of topological dimension consists of a notion of dimension of topological spaces (not necessarily *linear spaces*), which is invariant under homeomorphisms, and amounts to the usual notion of dimension in the standard cases such as, for example $\mathbb{R}^n$, or manifolds. Moreover, the topological dimension is always an integer number, while the Hausdorff dimension is not necessarily integer, and is not necessarily (in fact is not) invariant under homeomorphisms. There are different ways to define the topological dimension, that are however equivalent in the case of separable metric spaces (in particular of subsets of $\mathbb{R}^n$), not in the case of arbitrary topological spaces. We will define when $\dim(X) \le n$, thus $\dim(X) = n$ will be defined by the obvious equivalence

$$\dim(X) = n \iff \Big(\dim(X) \le n, \dim(X) \not\le n-1\Big) .$$

The first notion of dimension is due to Poincaré, and is the following

$\dim(\emptyset) = -1$,

$\dim(X) \le n$ if and only if for every $x \in X$ there exists a basis $\{U_i\}$ of neighbourhoods of x such that $\dim\big(\partial U_i\big) \le n-1$.

The idea of this definition is that for example in $\mathbb{R}^n$ every point has a basis of neighbourhoods (the balls centered at x), whose boundaries have dimension $n-1$. The second notion is due to Lebesgue. We only see the (simplified) version when the space is a compact metric space (recall that the compact metric spaces are separable).

$\dim(X) \le n$ if and only if for every $\varepsilon > 0$ there exists an open covering of X having mesh smaller than ε such that the intersection of $n+2$ elements of it is empty.

This definition in inspired by the observation that we can cover a closed interval with arbitrarily small intervals, such that any three of them have empty intersection, we can cover a closed square with arbitrarily small open squares, such that any four of them have empty intersection, and so on. Clearly, both the previous definitions are invariant under homeomorphisms. Note that it is simple to prove that, with any of the previous definitions, the dimension of any subset of $\mathbb{R}^n$ is $\le n$, but it is far from trivial that the dimension of $\mathbb{R}^n$ is precisely n.

The notions of Hausdorff and topological dimension are not unrelated. Namely, it is possible to prove that

$$\dim_H(X) \ge \dim(X) \tag{3.2.1}$$

for every separable metric space X. There is no standard definition of a fractal. However, one of the proposed definition is that a fractal is a set X such that in (3.2.1) the strict inequality holds.

3.3 Hausdorff Dimension of Fractals

In this Section we will evaluate the Hausdorff dimension of self-similar sets. Here, we will mean that a self-similar set is the unique nonempty compact Γ in $\mathbb{R}^\nu$ such that

$$\Phi(\Gamma) = \Gamma, \quad \Phi(A) := \bigcup_{i=1}^{k} \psi_i(A) \ \ \forall\, A \subseteq \mathbb{R}^\nu , \tag{3.3.1}$$

where ψ_i, $i = 1, ..., k$, $k \ge 2$, are contracting similarities in $\mathbb{R}^\nu$, that is, ψ_i are maps from $\mathbb{R}^\nu$ into itself such that

$$||\psi_i(x) - \psi_i(y)|| = \sigma_i ||x - y|| \quad \forall\, x, y \in \mathbb{R}^\nu, \quad 0 < \sigma_i < 1\,.$$

We also assume that the fixed points P_i of ψ_i are all different, that is $P_{i_1} \neq P_{i_2}$ if $i_1 \neq i_2$. Note that in many cases we have $\sigma_i = \sigma$ for every i, that is the contraction factors are all equal. Put in the sequel $\sigma_{max} = \max\{\sigma_i, i = 1, ..., k\}$, $\sigma_{min} = \min\{\sigma_i, i = 1, ..., k\}$. We have required $k \geq 2$, the case $k = 1$ being trivial: in fact, if $k = 1$ Γ is the fixed point of the unique contracting similarity. We will use in the sequel the following trivial fact : $\mathrm{diam}(\psi_i(A)) = \sigma_i \mathrm{diam}(A)$. In the sequel of this section we will denote by $\bar{\alpha}$ the unique positive real number satisfying

$$\sum_{i=1}^{k} \sigma_i^{\bar{\alpha}} = 1\,. \tag{3.3.2}$$

As the map $t \mapsto \sum_{i=1}^{k} \sigma_i^{\alpha}$ is continuous, strictly decreasing as $\sigma_i < 1$, and tends to 0 at $+\infty$, and to $+\infty$ at 0, then such an $\bar{\alpha}$ exists and is unique. Note that, if $\sigma_i = \sigma$ for all i, then (3.3.2) amounts to $k\sigma^{\bar{\alpha}} = 1$, that in turns amounts to

$$\bar{\alpha} = \log_\sigma \left(\frac{1}{k}\right) = \frac{\ln\left(\frac{1}{k}\right)}{\ln(\sigma)} = \frac{\ln(k)}{\ln\left(\frac{1}{\sigma}\right)}\,,$$

for example, in the case of the Cantor set, $\bar{\alpha} = \frac{\ln 2}{\ln 3}$; in the case of the Sierpinski Gasket, $\bar{\alpha} = \frac{\ln 3}{\ln 2}$; in the case of the Sierpinski Carpet, $\bar{\alpha} = \frac{\ln 8}{\ln 3}$. We will prove that $\bar{\alpha}$ defined by (3.3.2) is the Hausdorff dimension of Γ under a suitable condition (open set condition), which we will now introduce.

Open Set Condition (shortly O.S.C.): there exists a nonempty bounded subset $\hat{A}$ of $\mathbb{R}^\nu$ such that the sets $\psi_i(\hat{A})$, $i = 1, ..., k$ are mutually disjoint and contained in $\hat{A}$.

In order to prove the result on the Hausdorff dimension of Γ, we need some lemmas.

Lemma 3.3.1. *If $T : \mathbb{R}^\nu \to \mathbb{R}^\nu$ satisfies*

$$T(0) = 0, \quad ||T(x) - T(y)|| = ||x - y||\,, \tag{3.3.3}$$

then T is linear, thus is a linear isomorphism and a homeomorphism.

Proof. Putting $y = 0$ in the second equality in (3.3.3), and using the first, we get $||T(x)|| = ||x||$. In other words, T preserves the distance (and the norm). Hence, using the known formula $u \cdot v = \frac{1}{2}\big(||u||^2 + ||v||^2 - ||u - v||^2\big)$, we see that T also preserves the scalar product, namely

$$T(x) \cdot T(y) = \frac{1}{2}\big(||T(x)||^2 + ||T(y)||^2 - ||T(x) - T(y)||^2\big) = \frac{1}{2}\big(||x||^2 + ||y||^2 - ||x - y||^2\big) = x \cdot y$$

so that, if e_i, $i = 1, ..., \nu$ denotes the canonical basis, we have $e_i \cdot e_j = \delta_{i,j}$, hence $T(e_i) \cdot T(e_j) = \delta_{i,j}$, that is $T(e_i)$ form an orthonormal basis. We will now prove that, for any $a_i \in \mathbb{R}$, we have

$$T\Big(\sum_{i=1}^{\nu} a_i e_i\Big) = \sum_{i=1}^{\nu} a_i T(e_i)\,, \tag{3.3.4}$$

so that T is linear. We have $T\big(\sum_{i=1}^{\nu} a_i e_i\big) = \sum_{i=1}^{\nu} b_i T(e_i)$ for some b_i, namely $b_i = T\big(\sum_{i=1}^{\nu} a_i e_i\big) \cdot T(e_i)$, so that $b_i = \big(\sum_{i=1}^{\nu} a_i e_i\big) \cdot e_i = a_i$ and (3.3.4) is proved, and T is linear. By (3.3.3), T is one-to-one, thus a linear isomorphism. As any linear map between finite-dimensional linear spaces is continuous, both T and T^{-1} are continuous, thus T is a homeomorphism. ■

Corollary 3.3.2. *Every ψ_i is a homeomorphism from $\mathbb{R}^\nu$ onto $\mathbb{R}^\nu$, hence $\psi_i(\hat{A})$ is open.*

Proof. Put $T(x) := \dfrac{\psi_i(x) - \psi_i(0)}{\sigma_i}$. Since ψ_i is a contraction of factor σ_i, we see that T satisfies the hypothesis of Lemma 3.3.1, thus is a homeomorphism. As $\psi_i = \beta \circ T$, where $\beta(y) = \sigma y + \psi_i(0)$, thus clearly β is a homeomorphism, then ψ_i is a homeomorphism as well. ■

Recall that we put $E_{i_1,...,i_n} := \psi_{i_1,...,i_n}(E) = \psi_{i_1} \circ \cdots \circ \psi_{i_n}(E)$ for every subset E of $\mathbb{R}^\nu$. The definition of $\bar{\alpha}$ in (3.3.1) is motivated by the following lemma

Lemma 3.3.3. *For every bounded set E we have $h_{\bar{\alpha}}((E_{i_1,...,i_n}) : i_1, ..., i_n = 1, ..., k) = (diam(E))^{\bar{\alpha}}$.*

Proof. We have

$$h_{\bar{\alpha}}((E_{i_1,...,i_n}) : i_1, ..., i_n = 1, ..., k) = \sum_{i_1,...i_n=1}^{k} \big(\mathrm{diam} E_{i_1,...,i_n}\big)^{\bar{\alpha}} =$$

$$\sum_{i_1,...i_n=1}^{k} \big(\sigma_{i_1}^{\bar{\alpha}} \cdots \sigma_{i_n}^{\bar{\alpha}}\big)(\mathrm{diam} E)^{\bar{\alpha}} = (\mathrm{diam} E)^{\bar{\alpha}} \Big(\sum_{i_1=1}^{k} \sigma_{i_1}^{\bar{\alpha}}\Big) \cdots \Big(\sum_{i_n=1}^{k} \sigma_{i_n}^{\bar{\alpha}}\Big) = (\mathrm{diam} E)^{\bar{\alpha}}$$

where in the last equality we have used (3.3.2). ■

Note that, in view of (3.3.1), we have

$$\Phi^n(\Gamma) = \Gamma\,, \tag{3.3.5}$$

and, on the other hand,

$$\Phi^n(A) = \bigcup_{i_1,...,i_n=1}^{k} A_{i_1,...,i_n} \quad \forall\, A \subseteq \mathbb{R}^\nu\,. \tag{3.3.6}$$

Moreover, $\Phi(\hat{A}) \subseteq \hat{A}$, by the definition of $\hat{A}$, so that also $\Phi(\overline{\hat{A}}) \subseteq \overline{\hat{A}}$. It follows (see [P]) that $\Phi^n(\overline{\hat{A}})$ is a decreasing sequence of sets, and $\bigcap_{n=0}^{+\infty} \Phi^n(\overline{\hat{A}}) = \Gamma$, thus

$$\Gamma \subseteq \overline{\hat{A}}\,. \tag{3.3.7}$$

Lemma 3.3.4. *If the following hold*
i) $\exists\, c_1 > 0 \mid \forall\, \delta > 0\ \exists (U_i) \in \mathcal{S}_{\delta,\Gamma} \mid h_{\bar{\alpha}}((U_i)) \le c_1$.
ii) There exists $c_2 > 0$ *so that for every finite open covering* (U_i) *of* Γ *we have* $h_{\bar{\alpha}}((U_i)) \ge c_2$,
then $0 < H_{\bar{\alpha}}(\Gamma) < +\infty$, *thus* $\bar{\alpha} = dim_H(\Gamma)$.
Proof. By i) we have $H_{\bar{\alpha},\delta}(\Gamma) \le c_1$ for every δ, hence $H_{\bar{\alpha}}(\Gamma) \le c_1$. By ii), in view of (3.1.10), we have $H_{\bar{\alpha}}(\Gamma) \ge c_2$. ∎

Lemma 3.3.5. *For every self-similar set* Γ, *i) of Lemma 3.3.4 holds.*
Proof. By (3.3.5) and (3.3.6), $S := (\Gamma_{i_1,\dots,i_n}) : i_1, \dots, i_n = 1, \dots, k$, is a covering of Γ, and by Lemma 3.3.3, $h_{\bar{\alpha}}(S) = \big(\mathrm{diam}(\Gamma)\big)^{\bar{\alpha}} := c_1$. On the other hand,

$$\mathrm{diam}(\Gamma_{i_1,\dots,i_n}) = \sigma_{i_1} \cdots \sigma_{i_n} \mathrm{diam}(\Gamma) \le \sigma_{max}^n \mathrm{diam}(\Gamma) \le \delta\,,$$

thus $S \in \mathcal{S}_{\delta,\Gamma}$, for sufficiently large n. ∎

While i) of Lemma 3.3.4 is always valid, the validity of ii) essentially depends on O.S.C. We need some further notation. We put

$$D_n = \big\{(d_1, \dots, d_n) : d_1, \dots d_n = 1, \dots, k\big\}, \quad D_0 = \emptyset, \quad D = \bigcup_{n=0}^{+\infty} D_n\,.$$

If $d = (d_1, \dots., d_n)$, then, for ever $A \subseteq \mathbb{R}^\nu$ we put $A_d = A_{d_1,\dots,d_n}$, $A_\emptyset = A$, $\sigma_d = \sigma_{d_1} \cdots \sigma_{i_n}$. We also put $n = l(d)$ (the length of d), $d_{(m)} = (d_1, \dots, d_m)$ when $m \le n$.

Lemma 3.3.6. *If O.S.C. holds, and* $(i_1, \dots, i_n), (i'_1, \dots., i'_m) \in D$ *and there exists* $j \le \min\{n, m\}$ *such that* $i_j \ne i'_j$, *then* $\hat{A}_{i_1,\dots,i_n} \cap \hat{A}_{i'_1,\dots,i'_m} = \emptyset$.
Proof. We can and do assume $i_h = i'_h$ for every $h < j$, so that

$$\hat{A}_{i_1,\dots,i_n} = \psi_{i_1,\dots,i_{j-1}}\Big(\psi_{i_j}\big(\psi_{i_{j+1},\dots,i_n}(\hat{A})\big)\Big)\,,$$

$$\hat{A}_{i'_1,\dots,i'_m} = \psi_{i_1,\dots,i_{j-1}}\Big(\psi_{i'_j}\big(\psi_{i'_{j+1},\dots,i'_n}(\hat{A})\big)\Big)\,.$$

On the other hand, by O.S.C., $\psi_{i_j}\big(\psi_{i_{j+1},\dots,i_n}(\hat{A})\big) \subseteq \hat{A}_{i_j}$, $\psi_{i'_j}\big(\psi_{i'_{j+1},\dots,i'_m}(\hat{A})\big) \subseteq \hat{A}_{i'_j}$, and as the sets $\hat{A}_{i_j}$ and $\hat{A}_{i'_j}$ are disjoint by O.S.C. again, the sets $\psi_{i_j}\big(\psi_{i_{j+1},\dots,i_n}(\hat{A})\big)$ and $\psi_{i'_j}\big(\psi_{i'_{j+1},\dots,i'_m}(\hat{A})\big)$ are disjoint too. As the map $\psi_{i_1,\dots,i_{j-1}}$ is one-to-one, $\hat{A}_{i_1,\dots,i_n}$ and $\hat{A}_{i'_1,\dots,i'_m}$ are disjoint. ∎

Remark 3.3.7. if O.S.C. holds, we have $\hat{A}_{d_{(m)}} \supseteq \hat{A}_d$ for every $m \leq n := l(d)$. Indeed,

$$\hat{A}_{d_{(m)}} = \psi_{d_1} \circ \cdots \circ \psi_{d_m}(\hat{A}) \supseteq \psi_{d_1} \circ \cdots \circ \psi_{d_m}\Big(\psi_{d_{m+1}} \circ \cdots \circ \psi_{d_n}(\hat{A})\Big) = \hat{A}_d$$

as, by O.S.C., we have $\psi_{d_{m+1}} \circ \cdots \circ \psi_{d_n}(\hat{A}) \subseteq \hat{A}$. ∎

Theorem 3.3.8. *If O.S.C. holds, then ii) of Lemma 3.3.4 holds, and, consequently,* $0 < H_{\bar{\alpha}}(\Gamma) < +\infty$, *thus* $\bar{\alpha} = dim_H(\Gamma)$.

Proof. Let $(U_i), i = 1, ..., s$, be an open covering of Γ, where we mean that U_i are open in $\mathbb{R}^\nu$. We prove that this covering satisfies ii) of Lemma 3.3.4 for some $c_2 > 0$ independent of the covering. Let

$$\bar{R} := \mathrm{diam}(\hat{A}) > 0$$

let $B_{\bar{r}}(\bar{x}) \subseteq \hat{A}$. Such a ball exists as $\hat{A}$ is a nonempty open set. It follows

$$B_{\bar{r}\sigma_d}(\psi_d(\bar{x})) \subseteq \hat{A}_d \,. \tag{3.3.8}$$

In fact, if $y \in B_{\bar{r}\sigma_d}(\psi_d(\bar{x}))$, as, by Corollary 3.3.2, the map ψ_d, a composition of maps of $\mathbb{R}^\nu$ onto $\mathbb{R}^\nu$, is onto $\mathbb{R}^\nu$, there exists $x' \in \mathbb{R}^\nu$ such that $\psi_d(x') = y$. We have

$$\bar{r}\sigma_d > ||\psi_d(\bar{x}) - \psi_d(x')|| = \sigma_d||\bar{x} - x'||$$

hence, $x' \in B_{\bar{r}}(\bar{x})$, hence $y \in \psi_d\big(B_{\bar{r}}(\bar{x})\big) \subseteq \psi_d(\hat{A}) = \hat{A}_d$, and (3.3.8) is proved. We can assume

$$\mathrm{diam}(U_i) < \bar{R} \quad \forall\, i = 1, ..., s \,, \tag{3.3.9}$$

as, in the opposite case, $h_{\bar{\alpha}}((U_i)) \geq \bar{r}^{\bar{\alpha}}$. Put

$$\widetilde{W_i} = \big\{d \in D : \overline{\hat{A}}_d \cap U_i \neq \emptyset, \mathrm{diam}(\hat{A}_d) \leq \mathrm{diam}(U_i)\big\}\,,$$

$$W_i = \big\{d \in \widetilde{W_i} : d_{(n)} \notin \widetilde{W_i} \quad \forall\, n < l(d)\big\}\,.$$

For every $d \in W_i$ we have $l(d) = n > 0$, as, in the opposite case, $d = \emptyset$, $\hat{A}_d = \hat{A}$, $\mathrm{diam}(\hat{A}_d) \leq \mathrm{diam}(U_i)$ by the definition of $\widetilde{W_i}$, but this contradicts (3.3.9). We have $d_{(n-1)} \notin \widetilde{W_i}$, hence, as, by Remark 3.3.7 $\overline{\hat{A}_{d_{(n-1)}}} \cap U_i \neq \emptyset$, $\mathrm{diam}\big(\hat{A}_{d_{(n-1)}}\big) > \mathrm{diam}(U_i)$. As $\mathrm{diam}(\hat{A}_d) = \sigma_d\bar{R}$, $\mathrm{diam}(\hat{A}_{d_{(n-1)}}) = \sigma_{d_{(n-1)}}\bar{R}$, we have $\mathrm{diam}(\hat{A}_d) = \sigma_{d_n}\mathrm{diam}\big(\hat{A}_{d_{(n-1)}}\big)$. As a consequence,

$$\sigma_{min}\mathrm{diam}(U_i) < \mathrm{diam}(\hat{A}_d) = \bar{R}\sigma_d \leq \mathrm{diam}(U_i) \quad \forall\, d \in W_i \,. \tag{3.3.10}$$

Moreover,

$$\Big(d, d' \in W_i, d \neq d'\Big) \Rightarrow \hat{A}_d \cap \hat{A}_{d'} = \emptyset \,. \tag{3.3.11}$$

In fact, if (3.3.11) does not hold, then, by Lemma 3.3.6, we have $d_j = d'_j$ for every $j \leq \min\{l(d), l(d')\}$, and, if, for example $l(d') < l(d)$, we then have $d' = d_{(l(d'))}$, and this contradicts the definition of W_i. We now prove that there exists L independent of $i = 1, ..., s$ and of the covering such that

$$\#W_i \leq L \quad \forall\, i = 1, ..., s\,. \tag{3.3.12}$$

Pick a point $\tilde{x} \in U_i$. Then we have

$$\bigcup_{d \in W_i} B_{\bar{r}\sigma_d}(\psi_d(\bar{x})) \subseteq \bigcup_{d \in W_i} \overline{\hat{A}}_d \subseteq B_{2\mathrm{diam}(U_i)}(\tilde{x}) \tag{3.3.13}$$

In fact, if $x_i \in \overline{\hat{A}}_d \cap U_i$, $y \in \overline{\hat{A}}_d$. Then $||\tilde{x}-y|| \leq ||\tilde{x}-x_i||+||x_i-y|| \leq \mathrm{diam}(U_i)+\mathrm{diam}(\overline{\hat{A}}_d) \leq 2\,\mathrm{diam}(U_i)$ where we have used (3.3.10) in the last inequality. Moreover, by (3.3.8) and (3.3.11), the sets $B_{\bar{r}\sigma_d}(\psi_d(\bar{x}))$ are mutually disjoint. Hence, putting $\beta := \frac{\bar{r}}{\bar{R}}\sigma_{min}$, we have $\beta\mathrm{diam}(U_i) \leq \bar{r}\sigma_d$ by (3.3.10), thus, by (3.3.13) we have

$$\begin{aligned}\#W_i c_{0,\nu}\beta^\nu(\mathrm{diam}(U_i))^\nu &= \mu\Big(\bigcup_{d \in W_i} B_{\beta\mathrm{diam}(U_i)}\big(\psi_d(\bar{x})\big)\Big)\\ &\leq \mu\Big(B_{2\mathrm{diam}(U_i)}(\tilde{x})\Big) = c_{0,\nu}2^\nu(\mathrm{diam}(U_i))^\nu\,,\end{aligned}$$

where $c_{0,\nu}$ is defined in Lemma 1.1.1, thus $\#W_i \leq \dfrac{2^\nu}{\beta^\nu}$ and (3.3.12) is proved. Let

$$V_i = \big\{\hat{A}_d : d \in W_i\big\}\,,$$

$$(V_i)_{\bar{n}} := \big\{\hat{A}_d : \bar{n} = l(d), \exists\, n \leq \bar{n}, d_{(n)} \in W_i\big\} = \big\{\hat{A}_{d,i_{l(d)+1},\dots,i_{\bar{n}}} : d \in W_i\big\}$$

where $\bar{n}$ satisfies

$$\bar{n} > l(d) \quad \forall\, d \in \bigcup_{i=1}^{s} W_i\,, \tag{3.3.14}$$

$$\bar{R}\sigma_{max}^{\bar{n}} < \min_{i=1,\dots,s} \mathrm{diam}(U_i)\,. \tag{3.3.15}$$

Putting $\hat{A}_{D_{\bar{n}}} = \big\{\hat{A}_{i_1,\dots,i_{\bar{n}}} : i_1, ..., i_{\bar{n}} = 1, ..., k\big\}$, we have

$$\hat{A}_{D_{\bar{n}}} = \bigcup_{i=1}^{s}(V_i)_{\bar{n}}\,. \tag{3.3.16}$$

In fact, let $\hat{A}_d \in \hat{A}_{D_{\bar{n}}}$, let $x \in \Gamma_d$, thus $x \in \overline{\hat{A}}_d$ by (3.3.7). As (U_i) is a covering of Γ, there exists $i = 1, ..., s$ such that $x \in U_i$. Then, $\overline{\hat{A}}_d \cap U_i \neq \emptyset$, thus, by Remark 3.3.7,

$$\overline{\hat{A}_{(d_n)}} \cap U_i \neq \emptyset\,. \tag{3.3.17}$$

Moreover, $\mathrm{diam}(\hat{A}_d) < \mathrm{diam}(U_i)$, as

$$\mathrm{diam}(\hat{A}_d) = \sigma_d \mathrm{diam}(\hat{A}) = \bar{R}\sigma_d \le \bar{R}\sigma_{max}^{\bar{n}} < \mathrm{diam}(U_i)$$

by (3.3.15). As $d_{(0)} = \emptyset$ and $\hat{A}_{\emptyset} = \hat{A}$, in view of (3.3.9), then there exists $n \le \bar{n}$ such that $\mathrm{diam}(\hat{A}_{d_{(n)}}) \le \mathrm{diam}(U_i)$, but $\mathrm{diam}(\hat{A}_{d_{(n-1)}}) > \mathrm{diam}(U_i)$. Hence, in view of (3.3.17), $d_{(n)} \in W_i$, thus $\hat{A}_d \in (V_i)_{\bar{n}}$, and (3.3.16) holds. We now have

$$h_{\bar{\alpha}}(V_i) \le \#W_i (\mathrm{diam} U_i)^{\bar{\alpha}}\,, \tag{3.3.18}$$

$$h_{\bar{\alpha}}(V_i)_{\bar{n}} = \sum_{d \in W_i} \sum_{i_{n+1},\dots,i_{\bar{n}}=1}^{k} \big(\mathrm{diam}(\hat{A}_{d,i_{n+1},\dots,i_{\bar{n}}})\big)^{\bar{\alpha}} =$$

$$\sum_{d\in W_i} h_{\bar{\alpha}}\big((\hat{A}_{d,i_{n+1},\dots,i_{\bar{n}}}) : i_{n+1},\dots,i_{\bar{n}} = 1,\dots,k\big) = \sum_{d\in W_i} (\mathrm{diam}(\hat{A}_d))^{\bar{\alpha}} = h_{\bar{\alpha}}(V_i)$$

where we have used Lemma 3.3.3 in the third equality. Hence, in view of (3.3.18) and (3.3.12) we have $h_{\bar{\alpha}}\big((U_i)\big) \ge \frac{1}{L}\sum_{i=1}^{s} h_{\bar{\alpha}}(V_i)$, hence

$$h_{\bar{\alpha}}\big((U_i)\big) \ge \frac{1}{L}\sum_{i=1}^{s} h_{\bar{\alpha}}((V_i)_{\bar{n}}) \ge \frac{1}{L} h_{\bar{\alpha}}(\hat{A}_{D_{\bar{n}}}) = \frac{1}{L}(\mathrm{diam}\hat{A})^{\bar{\alpha}}$$

where we have used (3.3.16) in the second inequality and Lemma 3.3.3 again in the equality, and the Theorem is proved. ∎

Remark 3.3.8. By the definition of $H_{\bar{\alpha}}$, It is easy to verify that $H_{\bar{\alpha}}(A_i) = (\sigma_i^{\bar{\alpha}}) H_{\bar{\alpha}}(A)$ for every $i = 1,\dots,k$ and for every subset A of $\mathbb{R}^{\nu}$. Hence,

$$\sum_{i=1}^{k} H_{\bar{\alpha}}(\Gamma_i) = \Big(\sum_{i=1}^{k} \sigma_i\Big)^{\bar{\alpha}} H_{\bar{\alpha}}(\Gamma) = H_{\bar{\alpha}}\big(\bigcup_{i=1}^{k} \Gamma_i\big)$$

where we have used (3.3.1), thus, we deduce that $H_{\bar{\alpha}}(\Gamma_i \cap \Gamma_j) = 0$ for every $i, j = 1,\dots,k$, $i \ne j$. ∎

Remark 3.3.9. For every set of $(a_i \in \{0,1\}) : i = 1,\dots,\nu$, we define the contracting similarity of factor $\frac{1}{2}$, $x \mapsto \frac{1}{2}x + \sum_{i=1}^{\nu} \frac{a_i}{2} e_i$. The self-similar set corresponding to this set of similarities is $[0,1]^{\nu}$. We have 2^{ν} similarities and $\sigma_i = \frac{1}{2}$, so that $\dim_H(\Gamma) = \bar{\alpha} = \frac{\ln(2^{\nu}}{\ln(2)} = \nu$. Moreover $0 < H'_{\bar{\alpha}}(\Gamma) < +\infty$. Since, as easily verified, H'_{ν} is invariant under translations, it is a multiple of Lebesgue measure, by a well known result, and this multiple is positive and finite. This is a result announced in Section 3.3.1, which allows us to define $H_{\nu} = K_{\nu} H'_{\nu}$ where the constant K_{ν} can be chosen so that H_{ν} amounts to Lebesgue measure. ∎