2. General Topology

2.1 Fixed Points of Contractions.

In the present chapter, we will see some results in general topology, and, in particular, some fixed points results. A fixed point theorem states that, under suitable conditions, a map f has a fixed point, that is a point x such that $f(x) = x$. In this Section, we will discuss the well known fixed point theorem for contractions in a complete metric spaces.

Theorem 2.1.1. *Let (X, d) be a complete metric space. Suppose a map $f : X \to X$ is a contraction in the sense that there exists $c \in]0, 1[$ such that*

$$d\big(f(x), f(y)\big) \le c\, d(x, y) \quad \forall\, x, y \in X\,. \tag{2.1.1}$$

Then, there exsts a unique $\bar{x} \in X$ such that $f(\bar{x}) = \bar{x}$. Moreover, for every $x \in X$ the sequence $f^n(x) := f \circ f \circ f \circ \cdots \circ f(x)$ where f is composed n times, tends to $\bar{x}$ as n tends to infinity.

Proof. The uniqueness is trivial. Suppose x_1, x_2 are fixed points of f. Then, $f(x_1) = x_1$, $f(x_2) = x_2$, thus $d(x_1, x_2) = d\Big(f(x_1), f(x_2)\Big) \le cd(x_1, x_2)$, hence $d(x_1, x_2) = 0$ and $x_1 = x_2$. To prove the existence, pick a point $x \in X$, and we will prove that the sequence $f^n(x)$ tends to a fixed point. We will prove that $f^n(x)$ is a Cauchy sequence. By (2.1.1) in fact we have

$$d\big(f^n(x), f^{n+1}(x)\big) = d\Big(f\big(f^{n-1}(x)\big), f\big(f^n(x)\big)\Big) \le cd\big(f^{n-1}(x), f^n(x)\big)$$

thus, by induction, $d\big(f^n(x), f^{n+1}(x)\big) \le c^n d\big(x, f(x)\big)$. Hence, if $n \le m$,

$$\begin{aligned} d\big(f^n(x), f^m(x)\big) \le d\big(f^n(x), f^{n+1}(x)\big) + \cdots + d\big(f^{m-1}(x), f^m(x)\big) \le \\ \big(c^n + \cdots + c^{m-1}\big) d\big(x, f(x)\big) \le \Big(\sum_{i=n}^{+\infty} c_i\Big) d\big(x, f(x)\big) \\ = \frac{c^n}{1-c} d\big(x, f(x)\big) \underset{n\to+\infty}{\longrightarrow} 0\,. \end{aligned}$$

It immediately follows that $f^n(x)$ is a Cauchy sequence. As the space X is assumed to be complete, then there exists $\bar{x} := \lim_{n\to+\infty} f^n(x)$. On the other hand, f is a Lipshitz map of constant c by hypothesis, thus it is continuous. Therefore,

$$f(\bar{x}) = f\big(\lim_{n\to+\infty} f^n(x)\big) = \lim_{n\to+\infty} f\big(f^n(x)\big) = \lim_{n\to+\infty} f^{n+1}(x) = \bar{x}$$

so that $\bar{x}$ is a fixed point of f, and the Theorem is proved. ▪

2.2. Simplexes

We recall that a convex combination of points $P_i, i = 1, ..., h$ in a linear space is a point of the form $Q = \sum_{i=1}^{h} t_i P_i$ with $t_i \geq 0$, $\sum_{i=1}^{h} t_i = 1$. In particular, the convex combinations of two points P_1 and P_2 are the points of the form $tP_1 + (1-t)P_2$ with $t \in [0, 1]$. We observe the following trivial fact: If $P_1, ..., P_h$ *lie in* $\mathbb{R}$, then we have

$$\min_{i=1,\dots,h} P_i \leq Q \leq \max_{i=1,\dots,h} P_i \tag{2.2.1}$$

for every convex combination $Q = \sum_{i=1}^{h} t_i P_i$ of P_i. In fact

$$\min_{i=1,\dots,h} P_i = \sum_{i=1}^{h} t_i \min_{i=1,\dots,h} P_i \leq \sum_{i=1}^{h} t_i P_i \leq \sum_{i=1}^{h} t_i \max_{i=1,\dots,h} P_i = \max_{i=1,\dots,h} P_i$$

We recall that a function f from a convex set C in a linear space with values in $\mathbb{R}$ is said to be convex if, whenever we are given $x_1, x_2 \in C$, $t \in [0, 1]$, we have

$$f\big(tx_1 + (1-t)x_2\big) \leq tf(x_1) + (1-t)f(x_2)\,.$$

As a consequence, whenever we are given $x_1, ..., x_h \in C$ and $t_1, ..., t_h \in [0, +\infty[$ such that $t_1 + \cdots + t_h = 1$, we have

$$f\Big(\sum_{i=1}^{h} t_i x_i\Big) \leq \sum_{i=1}^{h} t_i f(x_i)\,.$$

This can be easily proved by induction on h.

Lemma 2.2.1. *Let X be a linear normed space. Then, for every $x \in X$, the map defined on X by $y \mapsto ||x - y||$ is convex.*

Proof. We have to prove that for every $x_1, x_2 \in X$ and $t \in [0, 1]$ we have

$$||x - \big(tx_1 + (1-t)x_2\big)|| \leq t||x - x_1|| + (1-t)||x - x_2||\,.$$

This is a simple consequence of the triangular inequality. In fact

$$||x - \big(tx_1 + (1-t)x_2\big)|| = ||t(x - x_1) + (1-t)(x - x_2)|| \leq$$

$$||t(x - x_1)|| + ||(1-t)(x - x_2)|| = t||x - x_1|| + (1-t)||x - x_2||\,. \qquad \blacksquare$$

Recall that the *convex hull* of a subset A of a linear space, denoted by $co(A)$, is the intersection of all convex sets containing it, in other words, the smallest convex set containing it. For example, the convex hull of a set of two points is the segment-line of vertices the two points, the convex hull of three points is the triangle of vertices the three points. The convex hull of a set in a linear space can be characterized in terms of convex combinations.

Lemma 2.2.2. *If A is a nonempty subset of a linear space X, then*

$$co(A) = \left\{ \sum_{i=1}^{h} t_i P_i \quad : h = 1, 2, 3, ..., \ P_i \in A, t_i \geq 0, \sum_{i=1}^{h} t_i = 1 \right\}. \tag{2.2.2}$$

Proof. Let B be the set defined in the right-hand side of (2.2.2). It is immediate to verify that B is convex, that is, if $Q_1, Q_2 \in B$ and $\tau \in [0, 1]$, then $\tau Q_1 + (1 - \tau)Q_2 \in B$. Also, $A \subseteq B$, as, if $P \in A$, we can write P in the form of (2.2.2), putting $h = 1$, $P_1 = P$, $t_1 = 1$. It remains to prove that, given a convex subset C of X containing A, then C contains B. In other words, we have to prove that, if $Q = \sum_{i=1}^{h} t_i P_i$ as in (2.2.2), that is, $t_i \geq 0$, $\sum_{i=1}^{h} t_i = 1$, then $Q \in C$. We will prove this by induction on h. If $h = 1$, then $Q = P_1 \in A \subseteq C$. Suppose the inductive hypothesis holds for $h = \bar{h}$ and prove it holds for $h = \bar{h} + 1$ as well. Suppose

$$Q = \sum_{i=1}^{\bar{h}+1} t_i P_i$$

as in (2.2.2). Then, either $t_i = 0$ for all $i = 1, ..., \bar{h}$ and $Q = P_{\bar{h}+1} \in A \subseteq C$, or $\sum_{i=1}^{\bar{h}} t_i > 0$, and

$$Q = t' Q' + t_{\bar{h}+1} P_{\bar{h}+1}, \tag{2.2.3}$$

$$t' := \sum_{i=1}^{\bar{h}} t_i, \quad Q' := \sum_{i=1}^{\bar{h}} \tau_i P_i, \quad \tau_i := \frac{t_i}{\sum_{i=1}^{\bar{h}} t_i}.$$

Now, as $\sum_{i=1}^{\bar{h}} \tau_i = 1$, then $Q' \in C$ by the inductive hypothesis, and, as $t_{\bar{h}+1} = 1 - t'$ by the hypothesis in (2.2.2), and C is convex, by (2.2.3) $Q \in C$. ∎

Corollary 2.2.3. *If A is a set in a normed linear space X and $x \in X$, then for every $P \in co(A)$ there exists $Q \in A$ such that $||x - Q|| \geq ||x - P||$.*

Proof. Let $P = \sum_{i=1}^{h} t_i P_i$ be as in (2.2.2). By Lemma 2.2.1

$$||x - P|| = ||x - \sum_{i=1}^{h} t_i P_i|| \leq \sum_{i=1}^{h} t_i ||x - P_i|| \leq \max_{i=1,...,h} ||x - P_i||$$

where in the last inequality we have used (2.2.1), and the Corollary is proved. ∎

Remark 2.2.4. If, in particular, there exists a point in A of maximum distance from x, and this case clearly occurs if A is finite, then $||x - P|| \leq \max_{Q \in A} ||x - Q||$ for every $P \in co(A)$. ∎

We now recall the well-known result that a nonempty closed convex set in $\mathbb{R}^n$ has a unique point of minimum distance from a given point in $\mathbb{R}^n$. The result is true, more generally in Hilbert spaces, but the proof is more complicated. Note that the existence proof does not use the convexity of the set, while the convexity is essential for the uniqueness. In fact, the center of a sphere has the same distance from every point in the sphere.

Theorem 2.2.5. *Let $y \in \mathbb{R}^n$ and let C be a closed, convex nonempty subset of $\mathbb{R}^n$. Then there exists a unique $P \in C$ such that $||y - P|| = \min_{Q \in C} ||y - Q||$.*

Proof. Let $\phi(x) := ||x - y||$. Then, as well known, ϕ is continuous. Moreover, $\phi(x) \geq ||x|| - ||y||$, so that ϕ has a minimum on C, in that it is a continuous function defined on a closed subset of $\mathbb{R}^n$ satisfying $\phi(x) \underset{||x|| \to \infty}{\longrightarrow} +\infty$. For the uniqueness, note the following general inequality:

$$||\frac{x_1 + x_2}{2} - y|| < ||x_1 - y|| \quad \forall\, x_1, x_2 \in \mathbb{R}^n : x_1 \neq x_2, ||x_1 - y|| = ||x_2 - y|| \,. \tag{2.2.4}$$

There are different simple proofs of (2.2.4). Here, we merely note its geometrical interpretation. The segment-lines with vertices x_1, y, and x_2, y are the two equal edges of an isosceles triangle, and they are longer than the relative height. Now, if x_1 and x_2 are two different points in C of minimum distance from y, then, in view of (2.2.4), the point $\frac{x_1+x_2}{2}$, which lies in C too, as C is convex, has distance even smaller from y, a contradiction. ▪

Definition 2.2.6. *An h-simplex in $\mathbb{R}^n$, $h \geq 1$, is a set C of the form $C = co\big(\{P_0, P_1, ..., P_h\}\big)$, where P_0, P_1,..., P_h are points in $\mathbb{R}^n$.*

In view of Lemma 2.2.2, we can characterize C as the set of

$$Q = \sum_{i=0}^{h} t_i P_i, \quad t_i \geq 0, \sum_{i=0}^{h} t_i = 1 \,. \tag{2.2.5}$$

We say that the simplex is *regular* if $P_1 - P_0$,..., $P_h - P_0$ are linearly independent. It is important to note that in such a definition, we can replace P_0 with any P_j, that is, the following condition A_j

The points $P_i - P_j$, $i = 0, ..., h$, $i \neq j$ are linearly independent, with $j = 0, ..., h$ fixed

is independent of j. Of course, to prove this, it suffices to prove that $A_0 \Rightarrow A_1$, the other implications being completely analogous. Suppose A_0 holds and a linear combination $\sum_{i=0,2,\ldots,h} d_i(P_i - P_1)$ of $P_i - P_1$ is 0, and prove that all coefficients d_i are equal to 0, thus A_1 holds. Noting that $P_i - P_1 = (P_i - P_0) - (P_1 - P_0)$, we have

$$0 = \sum_{i=0,2,\ldots,h} d_i(P_i - P_1) = \Big(\sum_{i=2}^{h} d_i(P_i - P_0)\Big) - d(P_1 - P_0), \quad d := \sum_{i=0,2,\ldots,h} d_i \,.$$

As we have assumed A_0, thus the points $P_i - P_0$ are linearly independent, we have $d_i = 0$ for $i = 2, ..., h$, and $d = 0$, so that also $d_0 = d - \sum_{i=2}^{h} d_i = 0$. Hence $d_i = 0$ for every

i, as claimed. In the following, the simplexes will be always intended to be regular, unless differently specified. The points $P_0, ..., P_h$ are called *vertices* of the regular simplex $co\big(\{P_0, P_1, ..., P_h\}\big)$. We will denote by $V(C)$ the set of the vertices of a regular simplex C.

Remark 2.2.7 If we have a regular h-simplex C of vertices $P_0, ..., P_h$, then every point Q of C can be written as in (2.2.5) for unique $t_0, ..., t_h$, which depend continuously on Q. To see this, note that if X denotes the linear spaces generated by $P_1 - P_0, ..., P_h - P_0$, then every point in $x \in X$ can be written as

$$x = \sum_{i=1}^{h} p_i(x)(P_i - P_0)$$

for unique coefficients $p_i(x)$, and p_i (the "projections" on $P_i - P_0$) are linear, and thus continuous with respect to x. These are well known facts from linear algebra. Now, under the condition $\sum\limits_{i=0}^{h} t_i = 1$, the equality $Q = \sum\limits_{i=0}^{h} t_i P_i$ amounts to $Q - P_0 = \sum\limits_{i=1}^{h} t_i(P_i - P_0)$, which in turns, amounts to $Q - P_0 \in X$ and $t_i = p_i(Q - P_0)$ for $i = 1, .., h$. Of course, if (2.2.5) holds, then $t_0 = 1 - \sum\limits_{i=1}^{h} t_i$, so that the numbers t_i are unique, as claimed. The numbers t_i in (2.2.5) will be called the *coefficients* of Q, and h will be called the *dimension* of the simplex. If, more specifically, $h = n$, then the points $P_i - P_0$ form a basis of $\mathbb{R}^n$ and $X = \mathbb{R}^n$, and p_i are defined on all of $\mathbb{R}^n$. ∎

Proposition 2.2.8. *An h simplex C in $\mathbb{R}^n$ is compact. If C is regular and $h = n$, then the interior of C is nonempty; moreover, for every $Q \in C$, then Q is an interior point of C if and only if all coefficients of Q are positive.*

Proof. Let $P_0, ..., P_h$ be the vertices of C. The set

$$T = \Big\{(t_0, t_1, ..., t_h) \in \mathbb{R}^{h+1} : t_i \geq 0, \sum_{i=0}^{h} t_i = 1\Big\}$$

is closed and bounded in $\mathbb{R}^{h+1}$, thus compact. Then, as C is the image of the continuous map from T to $\mathbb{R}^n$ defined by $(t_0, ..., t_h) \mapsto \sum\limits_{i=0}^{h} t_i P_i$, then C is compact. If C is regular and $h = n$, then we get that $Q \in C$ if and only if

$$p_i(Q - P_0) \geq 0 \quad \forall\, i = 1, .., n, \; \sum_{i=1}^{n} p_i(Q - P_0) \leq 1\,. \tag{2.2.6}$$

Indeed, if $Q \in C$, and Q is written as in (2.2.5), then by the discussion in Remark 2.2.7, we have

$$p_i(Q - P_0) = t_i \geq 0, \quad \sum_{i=1}^{n} p_i(Q - P_0) = \sum_{i=1}^{n} t_i = 1 - t_0 \leq 1\,.$$

We now prove that, conversely, if (2.2.6) holds, then $Q \in C$. In fact, putting $t_i = p_i(Q-P_0)$ for $i = 1, ..., n$, and $t_0 = 1 - \sum_{i=1}^{n} t_i \geq 0$, we have $\sum_{i=0}^{n} t_i = 1$, so that, by the discussion in Remark 2.2.7 again, we have $Q = \sum_{i=0}^{n} t_i P_i \in C$. Summarizing,

$$C = P_0 + \left(\bigcap_{i=1}^{n} p_i^{-1}([0, +\infty[) \cap \left(\sum_{i=1}^{n} p_i \right)^{-1} (]-\infty, 1]) \right) \supseteq$$

$$P_0 + \left(\bigcap_{i=1}^{n} p_i^{-1}(]0, +\infty[) \cap \left(\sum_{i=1}^{n} p_i \right)^{-1} (]-\infty, 1[) \right).$$

The set E in the second line of last formula is clearly open, and is nonempty as contains the points Q with positive coefficients. In fact, for some $t_i > 0$ with $\sum_{i=0}^{n} t_i = 1$, we have $Q = \sum_{i=0}^{n} t_i P_i = P_0 + \sum_{i=1}^{n} t_i (P_i - P_0)$ with $t_i > 0$, $\sum_{i=1}^{n} t_i < 1$, and, on the other hand, $t_i = p_i(Q - P_0)$, so that $Q \in E$. If instead, at least one coefficient of Q, e.g., t_1, is 0, then $p_1(Q-P_0) = 0$, and for every $\varepsilon > 0$ the point $Q_\varepsilon := Q - \varepsilon(P_1 - P_0) \notin C$ as $p_1(Q_\varepsilon - P_0) = -\varepsilon$ (see (2.2.6)), hence Q is not in the interior of C. ■

We will call *interior points* of a regular h-simplex C in $\mathbb{R}^n$ the points of C having positive coefficients, even if $h < n$. In such a case, however, they are not interior points in the topological sense. Note that, if C and C' are two regular simplexes and $C \subseteq C'$, and at least one point of C is interior to C', then *every* interior point of C is interior to C', in particular $Bar(C)$ lies in the interior of C'. Indeed, in such a case, for every vertex P of C' at least one vertex of C has positive P-coefficient. Hence, the P-coefficient of every interior point of C is positive.

Proposition 2.2.9. *The diameter of a simplex C amounts to the maximum of the distance of its vertices.*

Proof. Let P, Q be two points of C. Then, by Corollary 2.2.3, there exists a vertex P_i of C such that $||P - Q|| \leq ||P - P_i||$. For the same reason, there exists a vertex P_j of C such that $||P - P_i|| \leq ||P_j - P_i||$. ■

The barycenter of the simplex C of vertices $P_0, ..., P_h$, denoted by $Bar(C)$, or also by $Bar(\{P_0, P_1, ..., P_h\})$, is by definition the point $\frac{1}{h+1} \sum_{i=0}^{h} P_i$. Such a point is a convex combination of P_i, thus it is an element of C. We have

Lemma 2.2.10. *With the above notation $||Bar(C) - P|| \leq \frac{h}{h+1} \max_{i,j=0,\ldots,h} ||P_i - P_j||$ for every $P \in C$.*

Proof. We have $||Bar(C) - P|| \leq ||Bar(C) - P_j||$ for some $j = 0, ..., h$ by Corollary 2.2.3. On the other hand,

$$||Bar(C) - P_j|| = ||\Big(\frac{1}{h+1}\sum_{i=0}^{h} P_i\Big) - P_j|| =$$

$$||\frac{1}{h+1}\sum_{i=0}^{h}(P_i - P_j)|| = ||\frac{1}{h+1}\sum_{i:i\neq j}(P_i - P_j)||$$

$$\leq \frac{1}{h+1}\sum_{i:i\neq j}||P_i - P_j|| \leq \frac{1}{h+1}\sum_{i:i\neq j}\Big(\max_{i,j=0,\ldots,h}||P_i - P_j||\Big) = \frac{h}{h+1}\max_{i,j=0,\ldots,h}||P_i - P_j||\,. \qquad \blacksquare$$

A k-face of an h-simplex of C, $k \leq h$ is a k-simplex whose set of vertices is a subset of the set of the vertices of C, and we say that the face is proper if $k < h$.

Remark 2.2.11. An element of C lies in a proper face if and only if at least one of its coefficients is 0, hence, in view of Prop. 2.2.8, the boundary of C is the union of its proper faces or also is the union of its $(h-1)$-faces, as every proper face is contained in an $(h-1)$-face. As a consequence, $\mathrm{Bar}(C)$ is contained in no proper face of C. $\blacksquare$

It is possible, and not very difficult, to prove that a regular n-simplex in $\mathbb{R}^n$ is homeomorphic to a closed ball in $\mathbb{R}^n$. However, we will never use this fact.

2.3 Barycentric Subdivisions and Sperner Lemma.

Given a regular simplex $C = co\Big(\{P_0, ..., P_h\}\Big)$, fixed in the following of this Section, we define

$$\mathcal{A}(C)(=\mathcal{A}) = \Big\{(A_0, A_1, ..., A_h) : \emptyset \neq A_0 \subsetneq A_1 \subsetneq A_2 \cdots \subsetneq A_k = \{P_0, ..., P_h\}\Big\}\,,$$

$$\mathcal{C}(C)(=\mathcal{C}) = \Big\{co\Big(\{Bar(A_0), Bar(A_1), ..., Bar(A_h)\}\Big) : (A_0, A_1, ..., A_k) \in \mathcal{A}\Big\}\,.$$

In other words, $\mathcal{C}(C)$ is the set of the simplexes obtained by taking the barycenters of a sequence of increasing faces of C. $\mathcal{C}$ is called the *barycentric subdivision of* C. Note that $Bar(A_0), Bar(A_1), ..., Bar(A_h)$ are points in C, so that every element of $\mathcal{C}(C)$ is contained in C. Moreover, it is not difficult to prove that every element of $\mathcal{C}(C)$ is regular, provided C is regular.

Remark 2.3.1. The elements of the set of the interiors of the (not necessarily proper) faces of the elements of $\mathcal{C}(C)$ are mutually disjoint. Note that the generic elements of $\mathcal{C}(C)$ is given by

$$co\Big(\{Bar(P_{u(0)})\}, Bar\{P_{u(0)}, P_{u(1)}\}, ..., Bar\{P_{u(0)}, P_{u(1)}, ..., P_{u(h)}\}\Big)\,,$$

where u is a permutation of the indices $0, ..., h$, and a face of it is given by

$$co\Big(\big\{Bar(P_{u(0)})\big\}, Bar\big\{P_{u(0)}, P_{u(1)}\big\}, ..., Bar\big\{P_{u(0)}, P_{u(1)}, ..., P_{u(k)}\big\}\Big)\,,$$

$k \leq h$. Now, if P is in the interior such a face, then

$$P = \sum_{i=0}^{k} c_i \frac{1}{i+1}\Big(\sum_{j=0}^{i} P_{u(j)}\Big)$$

with $c_i > 0$, and $\sum_{i=0}^{k} c_i = 1$. Then, clearly, putting $P = \sum_{i=0}^{h} d_i P_i$ where d_i are the coefficients of P, we have $d_{u(0)} > d_{u(1)} > \cdots > d_{u(k)} > 0 = d_{u(k+1)} = \cdots = d_{u(h)}$. This chain of inequalities univocally determines the dimension k of the face, and $u(0), u(1), ..., u(k)$, that is the face of the element of $\mathcal{C}(C)$. ∎

The *mesh* of a set of subsets of $\mathbb{R}^n$ is by definition the supremum of the diameters of those subsets. We have

Lemma 2.3.2. $mesh(\mathcal{C}(C)) \leq \dfrac{h}{h+1} diam(C)$.

Proof. We use the above notation and take $T \in \mathcal{C}(C)$. We have to prove that $\mathrm{diam}T \leq \frac{h}{h+1}\mathrm{diam}(C)$. Let $T = co\big(\{Bar(A_0), Bar(A_1), ..., Bar(A_h)\}\big)$. By Prop. 2.2.9, it suffices to prove that

$$||Bar(A_i) - Bar(A_j)|| \leq \frac{h}{h+1} diam(C)\,. \tag{2.3.1}$$

In (2.3.1) we can and do assume $i < j$. By definition, $Bar(A_i) \in co(A_i) \subseteq co(A_j)$, thus, by Lemma 2.2.10,

$$||Bar(A_i) - Bar(A_j)|| \leq \frac{h}{h+1} diam(co(A_j)) \leq \frac{h}{h+1} diam(C)$$

as $A_j \subseteq A_h$, hence $co(A_j) \subseteq co(A_h) = C$. ∎

We now define the m-barycentric subdivision $\mathcal{C}_m(C)$ inductively.

$$\mathcal{C}_0(C) = \{C\}, \quad \mathcal{C}_{m+1}(C) = \bigcup_{D \in \mathcal{C}_m(C)} \mathcal{C}(D)\,.$$

Corollary 2.3.3. *We have* $mesh\big(\mathcal{C}_m(C)\big) \leq \Big(\dfrac{h}{h+1}\Big)^m diam(C)$.

Proof. This follows at once, by induction on m, from Lemma 2.3.2. ∎

Remark 2.3.4. The construction of the barycentric subdivision implies that the restriction of the barycentric subdivision to a $(h-1)$-face F, i.e., the intersection of the simplexes

of the barycentric subdivision with F, is the barycentric subdivision of F, and similarly for the m-barycentric subdivision. Note that
i) A face F of an element of $\mathcal{C}(C)$ lies in an $(h-1)$-face of C if and only if it does not contain $Bar(C)$ as a vertex. In such a case we will say that F is a *boundary face* of $\mathcal{C}(C)$, in the opposite case that F is an *interior face* of $\mathcal{C}(C)$.
ii) The interior of the faces of the elements of $\mathcal{C}_m(C)$ are mutually disjoint. This can be proved by induction on m using Remark 2.3.1.
iii) By induction on m, we can also prove that the barycenter of any element of $\mathcal{C}_m(C)$ lies in the interior of C.
iv) As a consequence, the barycenter of an element of $\mathcal{C}_m(C)$ cannot belong to another element of $\mathcal{C}_m(C)$. ■

Lemma 2.3.5. *Let F be an $(h-1)$-face of an element of $\mathcal{C}_m(C)$. If F is contained in an $(h-1)$-face of C, then it is an $(h-1)$-face of exactly one element of $\mathcal{C}_m(C)$; if F is contained in no $(h-1)$-face of C, then it is an $(h-1)$-face of exactly two elements of $\mathcal{C}_m(C)$.*

Proof. This result has a clear geometrical interpretation. However, we are going to give a formal proof, which will turn to be a bit technical. Consider first the case $m = 1$. An element of $\mathcal{C}_1(C)$ has the form

$$D = co\big(\{B_0, B_1, ..., B_h\}\big), \quad B_j := Bar(A_j), \quad A_j := \{P_{u(0)}, P_{u(1)}, ..., P_{u(j)}\}$$

where u is a permutation of $\{0, 1, ..., h\}$. Note that $B_h = Bar(C)$. An $(h-1)$-face of it is given by removing one of the B_j, that is, $F = co\big(\{B_0, B_1, ...B_{j-1}, B_{j+1}, ..., B_h\}\big)$. If the index removed is h, then

$$F = co\big(\{B_0, B_1, ..., B_{h-1}\}\big) \subseteq co\big(\{P_{u(0)}, P_{u(1)}, ..., P_{u(h-1)}\}\big)$$

that is F is a boundary face of C. Moreover, D is the only element of $\mathcal{C}(C)$ whose F is a face. In fact, every element of $\mathcal{C}(C)$ contains $B_h = Bar(C)$ as a vertex. If the index removed j is different from h, then F contains $Bar(C)$, thus is an interior face of C. Moreover, as A_{j+1} is obtained by A_{j-1} adding two elements $P_{u(j)}$ and $P_{u(j+1)}$, then F is a face of two elements of $\mathcal{C}(C)$, namely those where A_j is replaced by a set obtained adding either $P_{u(j)}$ or $P_{u(j+1)}$ to the vertices of A_{j-1}. Thus, the Lemma is proved if $m = 1$.

If $m > 1$, assume the Lemma is true for $m-1$. By the hypothesis, F is an $(h-1)$-face of an element of $\mathcal{C}(C')$, where C' is an element of $\mathcal{C}_{m-1}(C)$. If F is an interior face of $\mathcal{C}(C')$, then it is a face of two elements of $\mathcal{C}(C')$ by what we have proved in the case $m = 1$, and, as it contains the barycenter of C', by Remark 2.3.4 iv) C' is the face of this only two elements of $\mathcal{C}_m(C)$. Moreover, by Remark 2.3.4 iii), it is contained in no $(h-1)$ face of C.

If, on the contrary, F is a boundary face of $\mathcal{C}(C')$, by the case $m = 1$, F is a face of exactly one element C_1 of $\mathcal{C}(C')$. Moreover, it is contained in a face F' of C' and $Bar(F')$ is a vertex of F. By the inductive hypothesis, if F' is contained in an $(h-1)$-face of C, then C' is the only elements of $\mathcal{C}_{m-1}(C)$ whose F' is a face, thus C_1 is the only elements of $\mathcal{C}_m(C)$ whose F is a face. If, instead F' is contained in no $(h-1)$-face of C, then there are exactly two elements of $\mathcal{C}_{m-1}(C)$ whose F' is a face, and one of these is C', and we call the other C''. Every element of $\mathcal{C}_m(C)$ whose F is a face is necessarily contained either in C' or in C'', and by the case $m = 1$ both C' and C'' contain exactly one element of $\mathcal{C}_m(C)$ whose F is a face. Moreover, as F' contains interior points of C, thus $Bar(F')$, which is a vertex of F, is an interior point of C, also F contains interior points of C, thus it is contained in no $(h-1)$-face of C. Thus, the inductive step is completed, and the Lemma is proved. ■

Theorem 2.3.6. (Sperner Lemma) *Let C be a regular h-simplex, $h \geq 1$, $m \geq 1$, let*

$$V := \bigcup_{T \in \mathcal{C}_m(C)} V(T), \quad W := V(C),$$

and let α be a map from V to W, such that, if Q belongs to a face F of C, then $\alpha(Q)$ is one of the vertices of F. Then there exists $T \in \mathcal{C}_m(C)$ such that $\alpha(T) = W$.

Proof. Note that the hypothesis on α means, for example, that if $Q \in W$, then $\alpha(Q) = Q$, if, more generally, $Q \in co\big(\{P_{j_0}, ..., P_{j_s}\}\big)$, then $\alpha(Q) = P_{j_k}$ for some $k = 0, ..., s$. Here $P_0, ..., P_h$ are the vertices of C. We will say that $T \in \mathcal{C}_M(C)$ is *normal* if $\alpha(T) = W$. The statement of this Theorem is that there exists at least one normal $T \in \mathcal{C}_m(C)$.

We will prove by induction on h a stronger result. Namely, the set of normal $T \in \mathcal{C}_m(C)$ is *odd*. If $h = 1$, then $W = \{P_0, P_1\}$. The elements of $\mathcal{C}_m(C)$ are $\{Q_{i-1}, Q_i\}$, with $i = 1, ..., 2^m$, and $Q_0 = P_0$, $Q_{2^m} = P_1$. This can be easily seen by induction on m. By hypothesis we have $\alpha(Q_0) = P_0$, $\alpha(Q_{2^m}) = P_1$. Also, $\{Q_{i-1}, Q_i\}$ is normal if and only if $\alpha(Q_{i-1}) \neq \alpha(Q_i)$. Thus, the number of normal $\{Q_{i-1}, Q_i\}$ is the number of times α changes value. As α attains different values at the extremum points Q_0 and Q_{2^m}, such a number must be odd. Suppose now the statement holds for $h-1$ and prove it holds for h. Let

$$R := \Big\{(T, F) : T \in \mathcal{C}_m(C), F \ (h-1)\text{-face of } T, \alpha\big(V(F)\big) = \{P_1, ..., P_h\}\Big\}.$$

We will evaluate the number of elements of R in two different ways. Define

$$A_1 := \Big\{F : \ \exists T \text{ such that } (T, F) \in R, F \subseteq co(\{P_1, ..., P_h\})\Big\},$$

$$A_2 := \Big\{F : \ \exists T \text{ such that } (T, F) \in R, F \nsubseteq co(\{P_1, ..., P_h\})\Big\}.$$

We have

$$\#R = \#A_1 + 2\#A_2 . \tag{2.3.2}$$

To prove (2.3.2), observe that, if $F \in A_2$, then F is contained in no $(h-1)$-face of C, as if

$$F \subseteq co(\{P_j : j \neq \bar{j}\}), \tag{2.3.3}$$

then by the hypothesis on α,

$$\alpha(V(F)) \subseteq \{P_j : j \neq \bar{j}\},$$

but by the definition of R, $\alpha(V(F)) = \{P_1, ..., P_h\}$, so that $\bar{j} = 0$. Then (2.3.3) contradicts the definition of A_2. Hence, by Lemma 2.3.5, every $F \in A_2$ is a face of two $T \in \mathcal{C}_m$ with $(T, F) \in R$. For a similar reason, every $F \in A_1$, F being contained in the $(h-1)$-face $co(\{P_1, ..., P_h\})$, is a face of one $T \in \mathcal{C}_m$ with $(T, F) \in R$. Now, (2.3.2) easily follows. Let now

$$B_1 = \Big\{T \in \mathcal{C}_m(C) : \alpha\big(V(T)\big) = \{P_0, P_1, P_2, ..., P_h\}\Big\},$$

$$B_2 = \Big\{T \in \mathcal{C}_m(C) : \alpha\big(V(T)\big) = \{P_1, P_2, ..., P_h\}\Big\}.$$

Every $T \in B_1$ has one face F such that $(T, F) \in R$, namely, as α is a bijection from $V(T)$ to W, the face having for vertices the vertices of T that α sends to $P_1, P_2, ..., P_h$. On the other hand, if $T \in B_2$, then α maps two vertices Q_1, Q_2 of T to the same element $\bar{P}$ of W, and the other vertices of T to mutually different (and different from $\bar{P}$) elements of W. Thus, T has exactly two faces F such that $(T, F) \in R$, namely those omitting one of the two vertices mapped to $\bar{P}$. It follows

$$\#R = \#B_1 + 2\#B_2 \,. \tag{2.3.4}$$

Now, the restriction of α to the vertices of the m-barycentric subdivision of $co\big(\{P_1, ..., P_h\}\big)$ clearly inherits the properties of α, thus by the inductive hypothesis, it has an odd number of normal simplexes. But such normal simplexes, by definition are the elements of A_1, and the normal elements of $\mathcal{C}_m(C)$ are by definition the elements of B_1. As A_1 has an odd number of elements, in view of (2.3.2) and (2.3.4), B_1 has an odd number of elements as well. ▪

2.4 Brouwer and Schauder Fixed Point Theorems.

In this section we prove some important fixed point theorems. Such theorems, unlike the fixed point theorem for contractions, state the existence, but not the uniqueness of the fixed point. More precisely, we will prove that every continuous map from a nonempty compact convex set in $\mathbb{R}^n$ (or more generally in a Banach space) into itself has a fixed point. The general result is known as *Schauder Theorem*, while the case where the convex set is a closed ball in $\mathbb{R}^n$ is called *Brouwer Theorem.*

Theorem 2.4.1. *Let D be a nonempty compact convex subset of $\mathbb{R}^n$. Let f be a continuous map from D into D. Then F has (at least) a fixed point.*

Proof. Suppose for the moment D is a regular n-simplex with vertices $P_0, ..., P_h$. By contradiction, suppose f has no fixed points. We can then easily construct a continuous map ϕ from D to ∂D, with the property that $\phi(x) = x$ if $x \in \partial D$. The point $\phi(x)$ can be defined as the intersection of the open half-line with origin at $f(x)$ and passing through x, with ∂D. This definition is correct as, by the assumption, $f(x) \neq x$. Put now

$$x = \sum_{i=0}^{n} c_i(x) P_i \quad \forall\, x \in D$$

where $c_i : D \to [0,1]$ are continuous and $\sum_{i=0}^{n} c_i(x) = 1$ (cf. Remark 2.2.7). Of course, for every $J \subseteq \{0,1,...,n\}$ we have

$$x \in co(\{P_j : j \in J\}) \iff c_j(x) = 0 \; \forall j \notin J\,. \tag{2.4.1}$$

Put $\gamma_i = c_i \circ \phi$. Then, $\phi(x) = \sum_{i=0}^{n} \gamma_i(x) P_i$ and at least one $\gamma_i(x)$ is 0 as $\phi(x) \in \partial D$ (see Remark 2.2.11). On the other hand, as $\sum_{i=0}^{n} \gamma_i(x) = \sum_{i=0}^{n} c_i(\phi(x)) = 1$, then $\max_{i=0,\dots,n} \gamma_i(x) \geq \frac{1}{n+1}$. By the uniform continuity of γ_i on D, there exists $\delta > 0$ such that, if

$$A \subseteq D, \mathrm{diam} A < \delta$$

then $|\gamma_i(x) - \gamma_i(y)| < \frac{1}{n+1}$ for every $x, y \in A$. Therefore, let $\bar{x} \in A$ and $\bar{i}$ be such that $\gamma_{\bar{i}}(\bar{x}) = 0$, then $\gamma_{\bar{i}}(x) < \frac{1}{n+1}$ for every $x \in A$. As a consequence, defining

$$\alpha(x) = P_{\hat{i}} : \gamma_{\hat{i}}(x) = \max\{\gamma_i(x), i = 0, ..., n\}, \hat{i} = 0, ..., n$$

for every $x \in D$, then,

$$\alpha(x) \neq P_{\bar{i}} \quad \forall x \in A\,. \tag{2.4.2}$$

Note that in the definition of α the maximum could be taken at different points, but clearly, this is not a problem as we can choose in some way one of the maximum points. Now, in view of Corollary 2.3.3, we can take the m-barycentric subdivision of D with m such that $\mathrm{mesh}\big(\mathcal{C}_m(D)\big) < \delta$. Such m exists by Corollary 2.3.3. Then α, restricted to $V := \bigcup_{T \in \mathcal{C}_m(D)} V(T)$ satisfies the hypothesis of the Sperner Lemma. In fact, if

$$x \in co(\{P_j : j \in J\}), \quad J \subsetneq \{0, ..., n\}\,,$$

then, by (2.4.1) and the definition of α we have $\alpha(x) = P_j$ for some $j \in J$. By the Sperner Lemma, then, there exists $A \in \mathcal{C}_m(D)$ such that α maps $V(A)$ onto $\{P_0, ..., P_n\}$. But, as $\mathrm{diam} A < \delta$, this contradicts (2.4.2), and the Theorem is proved in the case D is a regular n-simplex.

In the general case, By Prop. 2.2.8, we can find, by possibly translating a given regular n-simplex, a regular n-simplex having 0 as an interior point, i.e., containing a ball centered at 0 of radius $r > 0$. By multiplying it by $\frac{R}{r}$, we obtain a regular n-simplex S containing the ball B centered at 0 of radius R where R is such that $B \supseteq D$. Such ball exists as D is bounded. Thus $S \supseteq D$. Let $p : S \to D$ be the projection given by Theorem 2.2.5, that $p(x)$ is the point of D of minimum distance from x. It is well known (and easy to prove) that p is continuous. Now, let $g = f \circ p : S \to S$. Then g is continuous, and really maps S into S as, for every $x \in S$, we have $p(x) \in D$ and $f(p(x)) \in D \subseteq S$. Since we already proved the Theorem for S, g has a fixed point x, that is $f(p(x)) = x \in D$, so that $p(x) = x$ and $f(x) = x$. ∎

Theorem 2.4.2. *Let D be a nonempty convex compact subset of a linear normed space X. Then every continuous f from D to D has (at least) a fixed point.*

Proof. For every $\varepsilon > 0$ there exist $P_0, ..., P_h \in D$ such that

$$\min_{i=0,\ldots,h} ||x - P_i|| < \varepsilon \tag{2.4.3}$$

for every $x \in D$. This is a simple consequence of the compactness of D, for example the set of the balls of radius ε has a finite subcovering. Let $V := sp(\{P_i : i = 0, ..., h\})$. Then V inherits from X a norm equivalent to the euclidean norm, as in finite-dimensional spaces all norms are equivalent. Let $F := co(\{P_i : i = 0, ..., h\})$, so that, by Lemma 2.2.2, $F = \left\{ \sum_{i=0}^{h} t_i P_i,\ t_i \geq 0, \sum_{i=0}^{h} t_i = 1 \right\}$, and by the same proof as in Prop. 2.2.8, F is compact. Moreover, $F \subseteq D$ as D is convex. Note that F is not necessarily a regular simplex, in that the vectors $P_i - P_0$ are not necessarily linearly independent. We will now construct an "approximation of identity" from D to F, more precisely, a continuous map $\beta : D \to F$ such that

$$||\beta(x) - x|| < 2\varepsilon \quad \forall\, x \in D\,. \tag{2.4.4}$$

Let $\alpha : [0, +\infty[\to [0, +\infty[$ be continuous and such that $\alpha(t) \begin{cases} = 1 & \text{if } t \leq \varepsilon \\ = 0 & \text{if } t \geq 2\varepsilon \end{cases}$. Let $d_i(x) = ||x - P_i||$, and let $\delta_i := \alpha \circ d_i$, $\beta(x) = \sum_{i=0}^{h} \left(\frac{\delta_i(x)}{\sum_{i=0}^{h} \delta_i(x)} \right) P_i$. We have $\sum_{i=0}^{h} \delta_i(x) > 0$ by (2.4.3). Moreover, $\beta(x)$, as a convex combination of P_i, belongs to F. More precisely, as when $||x - P_i|| \geq 2\varepsilon$, then $\delta_i(x) = 0$, $\beta(x)$ is in fact a convex combination of those P_i having distance from x smaller than 2ε, thus, by Remark 2.2.4 we have $||\beta(x) - x|| < 2\varepsilon$ and (2.4.4) holds. Now, the map $f_\varepsilon := \beta \circ f$, restricted to F, is a continuous map from F to F. As F is a nonempty convex compact subset of a finite-dimensional space, where the topology the euclidean one, we can apply Theorem 2.4.1 and conclude that f_ε has a fixed point $x \in F \subseteq D$. We have

$$||x - f(x)|| = ||f_\varepsilon(x) - f(x)|| = ||\beta(f(x)) - f(x)|| \leq 2\varepsilon\,. \tag{2.4.5}$$

As the map $x \mapsto ||x - f(x)||$ is continuous from the compact set D with values to $\mathbb{R}$, it has a minimum. By (2.4.5) and the arbitrarity of ε, such a minimum is 0. Then there exists $\bar{x} \in D$ such that $||\bar{x} - f(\bar{x})|| = 0$, and thus f has a fixed point. ■

The hypothesis of convexity is essential in Theorems 2.4.1 and 2.4.2. In fact, if D is an annulus centered at the origin, then every nontrivial rotation has no fixed points on D. On the other hand, if $D \subseteq \mathbb{R}^n$ is nonempty compact and not convex but homeomorphic to a convex subset of $\mathbb{R}^n$, then every continuous map from D into itself has a fixed point, as clearly, such a property is invariant under homeomorphisms.