1. Notation

When we use a function f and we do not specify where it takes values, we intend it has values in $\mathbb{R}$. This for example in phrases like "Let f be a function defined on A".

μ will denote the Lebesgue measure.

In an integral we will sometimes omit the variable, for example $\int f$ stands for $\int f(x)\,dx$.

$B_r(x)$ will denote the open ball with center x and radius r. We will not specify the space; it will usually be $\mathbb{R}^n$, n depending on the context .

We will use "a.e." for "almost everywhere".

$C(X)$ will denote the set of continuous functions on a topological space X with values in $\mathbb{R}$, $C_c(X)$ will denote the set of continuous functions on a topological space X with values in $\mathbb{R}$, with compact support.

Similarly, if U is an open set in $\mathbb{R}^N$ and $n = 1, 2, 3, ..., \infty$, then $C^n(U)$ will denote the set of functions on U with values in $\mathbb{R}$, of class C^n, $C^n_c(U)$ will denote the set of functions on U with values in $\mathbb{R}$, of class C^n with compact support.

We will denote the scalar product of u and v by $u \cdot v$.

1. Partial Differential Equations

1.1 Examples. The Laplace Operator

Examples of P.D.E. (short for Partial Differential Equations).

$$\Delta u = 0 \ \text{ Laplace equation}$$

$$u_{tt} = c\Delta_x u, \ c > 0 \ \text{ Heat equation}$$

$$u_{tt} = c^2\Delta_x u, \ c > 0 \ \text{ Wave equation}$$

Recall that the Laplace operator (or Laplacian) Δ is defined by $\Delta u(x) = \displaystyle\sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2}$. We denote by $\Delta_x u$ the Laplace operator with respect to the variables x, in other words, when we have a function u of n space variables $x_1, ..., x_n$ and a time variable t, by definition, $\Delta_x u(x,t) = \displaystyle\sum_{i=1}^{n} \frac{\partial^2 u(x,t)}{\partial x_i^2}$. We now give a geometrical interpretation of the Laplace operator. We need a Lemma.

Lemma 1.1.1. *For every $k \in \mathbb{R}$ and positive integer n there exists a constant $c_{k,n} > 0$ such that, for every $\bar{x} \in \mathbb{R}^n$*

$$\int_{B_r(\bar{x})} ||x-\bar{x}||^k \, dx = c_{k,n} r^{k+n}$$

Proof. We have

$$\int_{B_r(\bar{x})} ||x-\bar{x}||^k \, dx = r^k \int_{B_r(\bar{x})} ||\frac{x-\bar{x}}{r}||^k \, dx =$$

$$r^k r^n \int_{B_r(\bar{x})} ||\frac{x-\bar{x}}{r}||^k \frac{1}{r^n}, dx = r^{k+n} \int_{B_1(0)} ||y||^k \, dy$$

where we use the variable change $y = \frac{x-\bar{x}}{r}$, with the factor change (given by the mod. of Jacobian determinant) equal to $\frac{1}{r^n}$. Now, it suffices to put $c_{k,n} = \int_{B_1(0)} ||y||^k \, dy$. ▪

Theorem 1.1.2. *We have*

$$\Delta(u)(\bar{x}) = \frac{1}{c'_n} \lim_{r \to 0} \frac{\frac{1}{\mu\left(B_r(\bar{x})\right)} \int_{B_r(\bar{x})} u(x) \, dx - u(\bar{x})}{r^2}$$

for every $u : \Omega \to \mathbb{R}$ of class C^2, where Ω is any domain in $\mathbb{R}^n$ and c'_n is a suitable positive constant only depending on n.

Proof. We use the Taylor expansion of order 2, and get

$$u(x) = u(\bar{x}) + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(\bar{x})(x_i - \bar{x}_i) + \sum_{i_1, i_2 = 1, \ldots, n, i_1 \neq i_2} \frac{\partial^2 u}{\partial x_{i_1} \partial x_{i_2}}(\bar{x})(x_{i_1} - \bar{x}_{i_1})(x_{i_2} - \bar{x}_{i_2})$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(\bar{x})(x_i - \bar{x}_i)^2 + R(x, \bar{x})$$

where $R(x, \bar{x})$ is $o(||x - \bar{x}||^2)$. Now, if we integrate over $B_r(\bar{x})$ (where r is chosen so that $\overline{B_r(\bar{x})} \subseteq \Omega$), by a symmetry argument, the integrals of the summands in the first two sums amount to 0, thus we get

$$\int_{B_r(\bar{x})} u(x) \, dx = \mu\big(B_r(\bar{x})\big) u(\bar{x}) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(\bar{x}) \int_{B_r(\bar{x})} (x_i - \bar{x}_i)^2 \, dx + \int_{B_r(\bar{x})} R(x, \bar{x}) \, dx \, . \quad (1.1.1)$$

Now, by another symmetry argument the integrals $\int_{B_r(\bar{x})} (x_i - \bar{x}_i)^2 \, dx$ are all equal, so that they amount to

$$\frac{1}{n} \sum_{i=1}^{n} \int_{B_r(\bar{x})} (x_i - \bar{x}_i)^2 \, dx = \frac{1}{n} \int_{B_r(\bar{x})} ||x - \bar{x}||^2 \, dx = \frac{c_{2,n}}{n} r^{2+n}$$

(by Lemma 1.1.1). On the other hand, it is easy to see that the last integral in (1.1.1) is $o(r^{2+n})$, by Lemma 1.1.1 again and the fact that the integrand there is $o(||x-\bar{x}||^2)$. In conclusion,

$$\int\limits_{B_r(\bar{x})} u(x)\,dx = \mu\big(B_r(\bar{x})\big)u(\bar{x}) + \frac{c_{2,n}}{2n} r^{2+n}\Delta u(\bar{x}) + o(r^{2+n}),$$

so that, by dividing by $\frac{c_{2,n}}{2n}r^{2+n}$, taking into account that $\mu\big(B_r(\bar{x})\big) = c_{0,n}r^n$, we get the Theorem. ∎

In view of the previous theorem, the Laplace operator at a point is a sort of rate of increment of the average of the function near the point with respect to the value of the function at the point. In other words, if the Laplace operator at a point is positive (resp. negative) then the average of the function near the point is greater (resp. smaller) than the function at the point. By this point of view, for example, the heat equation express the intuitive idea that the temperature tends to increase (resp. decrease) at the points where the temperature is smaller (resp. greater) than in the surrounding points. The Laplace equation for example, represents the temperature in a thermically stable situation.

1.2 Recalls of Integration Theory

In this section, I recall some known results from integration theory. We recall the notion of hypersurface in $\mathbb{R}^n$. We are given a finite set of maps $\phi_d : \bar{A}_d \to \mathbb{R}^n$, $d = 1, ..., k$, where each A_d is an open set in $\mathbb{R}^{n-1}$ such that its boundary has measure 0, and ϕ_d is of class C^1 on some open set containing $\bar{A}_d$, and the Jacobian matrix $J_{\phi_d}(x)$ of ϕ_d has rank equal to $n-1$ on $\bar{A}_d$. We also suppose that the sets $\phi_d(A_d)$ are mutually disjoint. In such a case we say that the set B

$$B := \bigcup_{d=1}^{k} B_d, \quad B_d = \phi_d(\bar{A}_d)$$

is a hypersurface. If $f : B \to \mathbb{R}$ is continuous, then

$$\int\limits_B f := \sum_{d=1}^{k} \int\limits_{B_d} f\,,$$

$$\int\limits_{B_d} f = \int\limits_{A_d} \big(f \circ \phi_d\big)\ ||\psi_d||$$

where $\psi_d(x)$ is a vector in $\mathbb{R}^n$ having components equal (up to the sign) to the determinants of the square submatrices of rank $n-1$ of $J_{\phi_d}(x)$, more precisely, the h-components of $\psi_d(x)$ is the determinant of the matrix given by $\left(\frac{\partial(\phi_d)_i(x)}{\partial x_j}\right)$, $i \neq h$, multiplied by $(-1)^{h+n}$. Note that $\psi_d(x)$ is normal to B at $\phi_d(x)$. In fact, the columns of $J_{\phi_d}(x)$ form a basis of the tangent space to B at $\phi_d(x)$, and the scalar product of $\psi_d(x)$ and a fixed column of $J_{\phi_d}(x)$

amounts to the determinant of the matrix obtained by $J_{\phi_d}(x)$ adding an n-column equal to that column, as can be seen calculating the determinant in terms of the n-column. Such a determinant amounts to 0, as this matrix has two columns identical. We call $\nu_B(u)$ the normal unit vector to B at $u \in B$, for example its amounts to $\frac{\psi_d(x)}{||\psi_d(x)||}$, where $u = \phi_d(x)$. Note that this vector depends on the representation of the surface, up to a factor ± 1.

Lemma 1.2.1. *Let $\phi : U \to \mathbb{R}^n$ be a C^1 diffeomorphism from the open subset U of $\mathbb{R}^n$ to its image in $\mathbb{R}^n$. Let $\bar{A} \times [t_1, t_2] \subseteq U$, where A is an open set in $\mathbb{R}^{n-1}$ such that its boundary has measure 0. Let $\Sigma_t = \phi_t(\bar{A})$, where we put $\phi_t(x) = \phi(x,t)$. Put $r_t(\phi_t(x)) = \left|\frac{\partial}{\partial t}(\phi_t(x)) \cdot \nu_{\Sigma_t}(\phi_t(x))\right|$. Then*

$$\int_{\phi\left(\bar{A}\times[t_1,t_2]\right)} u = \int_{t_1}^{t_2} \Big(\int_{\Sigma_t} u r_t \Big)\, dt\,.$$

Proof. Define $\psi_t(x)$ as in the previous definition. Note that

$$r_t(\phi_t(x))||\psi_t(x)|| = \left|det(J_\phi(x,t))\right|. \tag{1.2.1}$$

In fact, by definition

$$r_t(\phi_t(x))||\psi_t(x)|| = \left|\frac{\partial}{\partial t}(\phi(x,t)) \cdot \psi_t(x)\right|.$$

Now, (1.2.1) easily follows from the expression of $det(J_\phi(x,t))$ in terms of the last column of the matrix $J_\phi(x,t))$, that is $\frac{\partial}{\partial t}(\phi(x,t))$. Using (1.2.1),

$$\int_{t_1}^{t_2} \Big(\int_{\Sigma_t} u r_t \Big)\, dt = \int_{t_1}^{t_2} \Big(\int_{\bar{A}} u(\phi(x,t)) \left|det(J_\phi(x,t))\right| \Big)\, dt$$

$$= \int_{\bar{A}\times[t_1,t_2]} u(\phi(x,t)) \left|det(J_\phi(x,t))\right| \Big)\, dx\, dt$$

$$\int_{\phi\left(\bar{A}\times[t_1,t_2]\right)} u(x,t)\, dx\, dt$$

where in the second equality we use the Fubini Theorem, and in the third the rule of variable change for multiple integrals. ∎

Corollary 1.2.2. *If u is continuous on $B_r(\bar{x})$ with values in $\mathbb{R}$, then*

$$\int_0^r \Big(\int_{\partial B_t(\bar{x})} u(x)\, dx \Big) dt = \int_{B_r(\bar{x})} u(x)\, dx\,.$$

Proof. We can split $\partial B_1(0)$ into finitely many pieces, represented in the form $\varphi(\bar{A})$. Then, put $\phi(x,t) = \bar{x} + t\varphi(x)$. Of course, $||\varphi(x)|| = 1$. Using the notation of Lemma 1.2.1, we have

$$r_t(\phi_t(x)) = \Big|\varphi(x) \cdot \nu_{\Sigma_t}(\phi_t(x))\Big| = 1$$

as $\nu_{\Sigma_t}(\phi_t(x))$ amounts to the unit normal vector to $\partial B_t(\bar{x})$ at $\phi(x,t)$, that is $\pm\varphi(x)$. By Lemma 1.2.1 we have

$$\int\limits_{\phi\left(\bar{A}\times[a,t]\right)} u = \int\limits_a^t \Big(\int\limits_{\Sigma_t} u\Big)\, dt\,,$$

and summing up the pieces,

$$\int\limits_a^r \Big(\int\limits_{\partial B_t(\bar{x})} u(x)\, dx\Big) dt = \int\limits_{B_r(\bar{x})\setminus B_a(\bar{x})} u(x)\, dx\,,$$

for every $a \in]0,r[$. Hence we get the result, taking the limit for $a \to 0$. Note that we have not integrated directly from 0 to r as the map ϕ would be singular at $(0,0)$. ■

Corollary 1.2.3. For every $x \in \mathbb{R}^n$, we have $\mu\big(B_r(x)\big) = \dfrac{r}{n} \displaystyle\int\limits_{\partial B_r(x)} 1$.

Proof. We have

$$\mu\big(B_r(x)\big) = \int\limits_{B_r(x)} 1 = \int\limits_0^r \Big(\int\limits_{\partial B_t(x)} 1\Big)\, dt =$$

$$\int\limits_0^r t^{n-1} \int\limits_{\partial B_1(x)} 1\, dt = \frac{r^n}{n} \int\limits_{\partial B_1(x)} 1 = \frac{r}{n} \int\limits_{\partial B_r(x)} 1\,.$$

■

We finally recall the statement of the divergence Theorem. Let u be a function of class C^1 from the open subset Ω of $\mathbb{R}^n$ to $\mathbb{R}^n$, and suppose A is an open set such that $\bar{A} \subseteq \Omega$, and the boundary of A is of class C^1, then

$$\int\limits_A \operatorname{div} u(x)\, dx = \int\limits_{\partial A} u(x) \cdot \nu_{\partial A}(x)\, dx$$

where $\nu_{\partial A}(x)$ (or simply $\nu(x)$) denotes the outward unit normal vector to $\nu_{\partial A}$ at x. The requirement on the regularity of ∂A can be relaxed. If we use the divergence Theorem, with u defined by $u_j = \begin{cases} 0 & \text{if } j \neq i \\ v & \text{if } j = i \end{cases}$ with v of class C^1 on Ω, we get

$$\int_A \frac{\partial v}{\partial x_i}(x)\,dx = \int_{\partial A} v(x)\nu_i(x)\,dx$$

We will often use such a form of the divergence Theorem.
In the following we will say that A is div-regular if both A and every set of the form $A \setminus B_r(x)$, with $\overline{B_r(x)} \subseteq A$ satisfy the hypothesis of the divergence theorem. For example every A with boundary of class C^1 is div-regular.

1.3 Harmonic Functions

Definition 1.3.1. *A function u defined from an open set Ω in $\mathbb{R}^n$ with values in $\mathbb{R}$ is said to be harmonic if it is of class C^2 and satisfies*

$$\Delta u = 0 \quad \textit{in } \Omega\,.$$

Examples of harmonic functions. Every linear (or affine) function, that is a function of the form $\sum_{i=1}^{n} a_i x_i + b$ is harmonic. The real and the imaginary part of a holomorphic f unction is harmonic (where of course we interpret a subset of the complex plane as a subset of $\mathbb{R}^2$). If $n = 1$, the harmonic functions are the affine functions. Indeed, in such a case, $\Delta u = u''$. We now see what radial functions are harmonic, in other words, we are searching for harmonic functions of the form $u(x) = f(||x||)$, defined on $\mathbb{R}^n \setminus \{0\}$. At first glance, it could appear to be unnatural to exclude 0, but the following calculations will show there are no nonconstant harmonic radial functions on all of $\mathbb{R}^n$. A simple calculation shows that, when $u(x) = f(||x||)$, then

$$\Delta u(x) = \frac{f''\big(||x||\big)||x||^3 + nf'\big(||x||\big)||x||^2 - f'\big(||x||\big)||x||^2}{||x||^3}\,.$$

Hence, u is harmonic on $\mathbb{R}^n \setminus \{0\}$ if and only if, whenever $x \neq 0$, we have $f''\big(||x||\big)||x||^3 + (n-1)f'\big(||x||\big)||x||^2 = 0$, that is, if and only if we have

$$f''(\rho)\rho^3 + (n-1)f'(\rho)\rho^2 = 0 \quad \forall\, \rho > 0$$

that, putting $z(\rho) = f'(\rho)$, amounts to $z'(\rho) = -(n-1)\frac{z(\rho)}{\rho}$. This is a standard case of a separable variables equation, and its solution is

$$z(\rho) = c\rho^{1-n}\,.$$

Hence, we have

$$f(\rho) = \begin{cases} c_1\rho^{2-n} + c_2 & \text{if } n > 2 \\ c_1 \ln(\rho) + c_2 & \text{if } n = 2 \end{cases}$$

and

$$u(x) = \begin{cases} c_1 \dfrac{1}{||x||^{n-2}} + c_2 & \text{if } n > 2 \\ c_1 \ln(||x||) + c_2 & \text{if } n = 2 \end{cases}$$

Note that in particular, the gravitational potential generated by a point-mass, which we suppose located at the origin, is a harmonic function, and, for symmetry reasons, has the form $f\big(||x||\big)$, hence, by the previous discussion, has the form $c_1 \frac{1}{||x||}$ as the actual space has dimension 3. In this case $c_2 = 0$, for example as we expect that the potential goes to 0 at infinity. Let now

$$\psi(\rho) = \begin{cases} \frac{1}{2-n}\rho^{2-n} & \text{if } n > 2 \\ \ln(\rho) & \text{if } n = 2 \end{cases}$$

so that

$$\psi'(\rho) = \rho^{1-n}\,.$$

PUT FOR THE FOLLOWING OF THIS CHAPTER

$$\overline{\psi}(\rho) = \bar{d}_n \psi(\rho), \quad \bar{d}_n := \Big(\int_{\partial B_1(0)} 1\Big)^{-1},$$

$$\overline{\phi}(x) = \overline{\psi}(||x||)\,.$$

Note that, by a homogeneity argument, the $(n-1)$-dimensional measure of the sphere centered at $\bar{x}$ of radius r, $\partial B_r(\bar{x})$ is given by

$$\int_{\partial B_r(\bar{x})} 1 = \frac{1}{\bar{d}_n} r^{n-1}\,. \tag{1.3.1}$$

By the previous considerations, $\overline{\phi}$ is harmonic on $\mathbb{R}^n \setminus \{0\}$, and the choice of the constant $\bar{d}_n$ is motivated by the following lemma.

Lemma 1.3.2.
i) We have $\operatorname{grad}\overline{\phi}(x) = \overline{\psi}'(||x||)\frac{x}{||x||}$ *for every* $x \in \mathbb{R}^n \setminus \{0\}$.
ii) $\int_{\partial B_\rho(\bar{x})} \operatorname{grad}\overline{\phi}(x-\bar{x}) \cdot \nu = 1$ *for every* $\rho > 0$ *and for every* $\bar{x} \in \mathbb{R}^n$.
Proof. We have

$$\frac{\partial \overline{\phi}(x)}{\partial x_i} = \overline{\psi}'(||x||)\frac{x_i}{||x||}$$

and i) easily follows. Also, $\overline{\psi}'(\rho) = \bar{d}_n \rho^{1-n}$. Hence, as $\nu_{\partial B_\rho(\bar{x})}(x) = \frac{x-\bar{x}}{||x-\bar{x}||}$, we easily get

$$\operatorname{grad}\overline{\phi}(x-\bar{x}) \cdot \nu_{\partial B_\rho(\bar{x})} = \bar{d}_n ||x-\bar{x}||^{1-n} \tag{1.3.2}$$

therefore,

$$\int\limits_{\partial B_\rho(\bar{x})} \operatorname{grad}\overline{\phi}(x-\bar{x})\cdot\nu =$$

$$\int\limits_{\partial B_\rho(\bar{x})} \bar{d}_n||x-\bar{x}||^{1-n} =$$

$$\bar{d}_n\rho^{1-n}\int\limits_{\partial B_\rho(\bar{x})} 1 = 1\,.$$

We have used (1.3.1) in the last equality. ∎

In the following Ω will always denote a nonempty open connected subset of $\mathbb{R}^n$.

Lemma 1.3.3. *Suppose A is div-regular, $\overline{A}\subseteq\Omega$. Suppose u and v of class C^2 on Ω. Then,*

$$\int\limits_A \Big(\operatorname{grad}u\cdot\operatorname{grad}v + v\Delta u\Big) = \int_{\partial A} v\operatorname{grad}u\cdot\nu\,. \tag{1.3.3}$$

Proof. It suffices to apply the divergence Theorem to the function $v\operatorname{grad}u$. ∎

Corollary 1.3.4. *If u is harmonic on Ω and A is as above, then*

$$\int_{\partial A}\operatorname{grad}u\cdot\nu \,=0\,.$$

Proof. It suffices to put $v=1$ in Lemma 1.3.3 and to observe that the integral on the left-hand side amounts to 0. ∎

Corollary 1.3.5. *If u, v and A are as in Lemma 1.3.3, then*

$$\int\limits_A u\Delta v - v\Delta u = \int_{\partial A}\Big(u\operatorname{grad}v - v\operatorname{grad}u\Big)\cdot\nu\,.$$

Proof. We use (1.3.3) and its analog obtained exchanging u and v

$$\int\limits_A \Big(\operatorname{grad}v\cdot\operatorname{grad}u + u\Delta v\Big) = \int_{\partial A} u\operatorname{grad}v\cdot\nu$$

and then we take the difference. ∎

1.4 Mean Value Properties and Related Topics

Lemma 1.4.1. *If u is harmonic on Ω and $\overline{B_r(\bar{x})} \subseteq \Omega$, then*

$$u(\bar{x}) = \frac{\bar{d}_n}{r^{n-1}} \int\limits_{\partial B_r(\bar{x})} u\,. \tag{1.4.1}$$

Proof. We use Corollary 1.3.5 with $A := B_r(\bar{x}) \setminus B_{\bar{r}}(\bar{x})$ where $0 < \bar{r} < r$, and $v(x) = \overline{\phi}(x - \bar{x})$. Note that we cannot use $A = B_r(\bar{x})$, as v would be not defined there, thus we have to remove a small ball around $\bar{x}$. As u and v are harmonic, the left-hand side in formula in Corollary 1.3.5 amounts to 0. Therefore,

$$\int\limits_{\partial B_r(\bar{x})} \Big(u \operatorname{grad} v - v \operatorname{grad} u\Big) \cdot \nu = \int\limits_{\partial B_{\bar{r}}(\bar{x})} \Big(u \operatorname{grad} v - v \operatorname{grad} u\Big) \cdot \nu\,.$$

We have used the fact that $\partial\big(B_r(\bar{x}) \setminus B_{\bar{r}}(\bar{x})\big) = \partial B_r(\bar{x}) \cup \partial B_{\bar{r}}(\bar{x})$ and the fact that the outer unit normal to $\partial\big(B_r(\bar{x}) \setminus B_{\bar{r}}(\bar{x})\big)$ at the points of $\partial B_{\bar{r}}(\bar{x})$ is the opposite of the outer unit normal to $\partial B_{\bar{r}}(\bar{x})$ at the same points. On the other hand, using Corollary 1.3.4 and the fact that, in our case, v is constant at the components of the boundary, $\int\limits_{\partial B_r(\bar{x})} v \operatorname{grad} u \cdot \nu = \int\limits_{\partial B_{\bar{r}}(\bar{x})} v \operatorname{grad} u \cdot \nu = 0$. Hence,

$$\int\limits_{\partial B_r(\bar{x})} u \operatorname{grad} v \cdot \nu = \int\limits_{\partial B_{\bar{r}}(\bar{x})} u \operatorname{grad} v \cdot \nu\,. \tag{1.4.2}$$

By (1.3.2) we have for $\rho = r, \bar{r}$

$$\int\limits_{\partial B_\rho(\bar{x})} u \operatorname{grad} v \cdot \nu = \int\limits_{\partial B_\rho(\bar{x})} \bar{d}_n\ \rho^{1-n} u \tag{1.4.3}$$

Hence, in view of (1.4.2),

$$\int\limits_{\partial B_r(\bar{x})} \bar{d}_n\ r^{1-n} u = \int\limits_{\partial B_{\bar{r}}(\bar{x})} \bar{d}_n\ \bar{r}^{1-n} u \tag{1.4.4}$$

The integral in the right-hand side tends to $u(\bar{x})$ as $\bar{r} \to 0$. Indeed, by (1.3.1) we have $u(\bar{x}) = \int_{\partial B_{\bar{r}}(\bar{x})} \bar{d}_n\ \bar{r}^{1-n} u(\bar{x})$, hence, for every $\varepsilon > 0$ we have

$$\left|u(\bar{x}) - \int_{\partial B_{\bar{r}}(\bar{x})} \bar{d}_n\ \bar{r}^{1-n} u\right| \leq$$

$$\int_{\partial B_{\bar{r}}(\bar{x})} \bar{d}_n\ \bar{r}^{1-n} \Big|u - u(\bar{x})\Big| \leq$$

$$\int_{\partial B_{\bar r}(\bar x)} \varepsilon \bar{d}_n \; \bar{r}^{1-n} = \varepsilon$$

for sufficiently small $\bar r$, by (1.3.1) again. Taking the limit for $\bar r \to 0$, we get the Lemma. ∎

Remark 1.4.2. In view of (1.4.3), (1.4.1) amounts to

$$u(\bar x) = \int_{\partial B_r(\bar x)} u(x) \operatorname{grad} \overline{\phi}(x - \bar x) \cdot \nu(x)\, dx \qquad ∎$$

Note that, in view of (1.3.1), Lemma 1.4.1 states that u takes at a point $\bar x$ the average of the values on a sphere centered at $\bar x$. The following theorem, simple consequence of the previous lemma, states that u takes at $\bar x$ the average of the values on a ball centered at $\bar x$. Such results are called mean value properties.

Theorem 1.4.3 (mean value Theorem). *If u is harmonic on Ω and $\overline{B_r(\bar x)} \subseteq \Omega$, then*

$$u(\bar x) = \frac{1}{\mu(B_r(\bar x))} \int\limits_{B_r(\bar x)} u\,. \tag{1.4.5}$$

Proof. By Lemma 1.4.1 and (1.3.1) we have

$$u(\bar x) \int\limits_{\partial B_t(\bar x)} 1 = \int\limits_{\partial B_t \bar x)} u$$

for every $t \in]0, r[$, hence, in view of Corollary 1.2.2, we have

$$\mu(B_r(\bar x)) u(\bar x) = u(\bar x) \int_0^r \Big(\int\limits_{\partial B_t(\bar x)} 1 \Big)\, dt = \int_0^r \Big(\int\limits_{\partial B_t \bar x)} u \Big)\, dt = \int\limits_{B_r(\bar x)} u$$

and we conclude. ∎

Theorem 1.4.4. (Strong maximum principle). *If u is defined on $\overline{\Omega}$, continuous on $\overline{\Omega}$ and harmonic on Ω, and u takes its maximum or its minimum at a point of Ω (that is at a point in the interior, not in the boundary, of Ω), then u is constant on $\overline{\Omega}$.*

Proof. Let $E := \big\{ x \in \Omega : u(x) = \max_{\overline{\Omega}} u \big\}$. This set is clearly closed. Moreover, as a consequence of the mean value theorem, it is also open. Indeed, if $\bar x \in E$, take $r > 0$ such that $\overline{B_r(\bar x)} \subseteq \Omega$. Then, by Theorem 1.4.3, we have

$$\int\limits_{B_r(\bar x)} u(\bar x) = u(\bar x) \int\limits_{B_r(\bar x)} 1 = \int\limits_{B_r(\bar x)} u\,.$$

Therefore $\displaystyle\int_{B_r(\bar{x})} \Big(u(\bar{x}) - u\Big) = 0$, and as the integrand is continuous, and nonnegative as $\bar{x} \in E$, then $u(\bar{x}) - u(x) = 0$ for every $x \in \overline{B_r(\bar{x})}$, and E is open. If u takes its maximum at a point of Ω, then $E \neq \emptyset$, and by the connectedness of Ω, $E = \Omega$, hence u is constant on Ω, and by a continuity argument, also on $\overline{\Omega}$. A similar argument works if u takes its minimum at a point of Ω. ∎

Corollary 1.4.5. *Suppose Ω is bounded, and u is defined on $\overline{\Omega}$, continuous on $\overline{\Omega}$ and harmonic on Ω. Then*

$$\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u \quad \forall x \in \overline{\Omega}\,. \tag{1.4.6}$$

Proof. If u is constant on $\overline{\Omega}$, then the result is trivial. In the opposite case, then there exist maximum and minimum of u on $\overline{\Omega}$, as $\overline{\Omega}$ is compact, and they are attained at points in $\partial\Omega$ by Theorem 1.4.4. ∎

The statement of Corollary 1.4.5 is called *(weak) maximum principle.* In fact, Theorem 1.4.4 is a stronger version of such a statement, in that it states that the inequalities in (1.4.6) hold and moreover are strict unless u is constant. Note also that, if Ω is not bounded, then the maximum and minimum in (1.4.6) need not exist. However, the analog of (1.4.6) with inf in place of min and sup in place of max, is in general false. It suffices to take $\Omega = \mathbb{R}\times]0, +\infty[$ and $u(x) = x_2$.

We will say that a continuous function from Ω to $\mathbb{R}$ has the mean value property if (1.4.5) holds whenever $\overline{B_r(\bar{x})} \subseteq \Omega$.

Remark 1.4.6. The strong maximum principle holds in fact for functions satisfying the mean value property. Indeed, it is the only property of harmonic functions used in the proof. It suffices in fact, that the mean value property holds *locally*, in the sense that, for every $\bar{x} \in \Omega$ there exists $\bar{r} > 0$ such that (1.4.5) holds when $0 < r < \bar{r}$. Moreover, this amounts to the fact that, for every $\bar{x} \in \Omega$ there exists $\bar{r} > 0$ such that (1.4.1) holds when $0 < r < \bar{r}$. In fact, the proof of the mean value Theorem shows that (1.4.1) and (1.4.5) are equivalent for sufficiently small r. ∎

Corollary 1.4.7. *If Ω is bounded and u_1 and u_2 are functions defined on $\overline{\Omega}$, continuous on $\overline{\Omega}$ and harmonic on Ω, such that $u_1 = u_2$ on $\partial\Omega$, then $u_1 = u_2$ on $\overline{\Omega}$.*

Proof. Clearly, $u_1 - u_2$ is harmonic on Ω, and amounts to 0 on $\partial\Omega$, thus, by the maximum principle, amounts to 0 on $\overline{\Omega}$. ∎

Theorem 1.4.3 states that every harmonic function has the mean value property. We will now see that also the converse holds, in other words, every (continuous, not necessarily C^2) function satisfying the mean value property is harmonic.

Theorem 1.4.8. *If u is a continuous function from Ω to $\mathbb{R}$ satisfying the mean value property, then u is harmonic on Ω.*

Proof. Note that, in view of Theorem 1.1.2, we simply need to prove that u is of class C^2. First of all, we prove that if u is a continuous function satisfying the mean property, then so do its partial derivatives. Let $\bar{x} \in \overline{\Omega}$. As the balls with radius r have all the same measure, we have

$$\frac{u(\bar{x}+te_i)-u(\bar{x})}{t} = \frac{1}{t\mu(B_r(\bar{x}))}\Big(\int\limits_{B_r(\bar{x}+te_i)} u - \int\limits_{B_r(\bar{x})} u\Big) =$$

$$\frac{1}{t\mu(B_r(\bar{x}))}\Big(\int\limits_{B_r(\bar{x}+te_i)\setminus B_r(\bar{x})} u - \int\limits_{B_r(\bar{x})\setminus B_r(\bar{x}+te_i)} u\Big) \tag{1.4.7}$$

for sufficiently small r and t. Now, suppose for simplicity $t>0$ and $i=n$, and evaluate the first integral in (1.4.7). Write $x \in \mathbb{R}^n$ in the form $x=(y,z)$, $y \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}$, $\bar{x}=(\bar{y},\bar{z})$. Let

$$\beta(y) = \sqrt{r^2-||y||^2}\,,$$
$$A_t = \big\{(y,z) : y \in \mathbb{R}^{n-1}, z > t\big\}\,,$$
$$C_t = \big\{\ (y,z) : y \in \overline{B_r(\bar{y})},\ \ z \in \,]\bar{z}+\beta(y-\bar{y}), \bar{z}+t+\beta(y-\bar{y})]\ \big\}\,,$$
$$D_t = \Big\{(y,z) : \bar{z} < z \le \bar{z}+t, y \in \overline{B_r(\bar{y})} \setminus \overline{B_{\sqrt{r^2-t^2}}(\bar{y})}\Big\}\,.$$

Then, it is easy to verify that, for any $r>0$, if $0<t<r$ we have

$$\Big(\overline{B_r(\bar{x}+te_i)} \setminus \overline{B_r(\bar{x})}\Big) \cap A_{\bar{z}+t} = C_t \cap A_{\bar{z}+t}\,, \tag{1.4.8}$$

$$\Bigg(\Big(\overline{B_r(\bar{x}+te_i)} \setminus \overline{B_r(\bar{x})}\Big) \setminus A_{\bar{z}+t}\Bigg) \cup \big(C_t \setminus A_{\bar{z}+t}\big) \subseteq D_t\,, \tag{1.4.9}$$

$$\frac{1}{t}\mu\big(D_t\big) \underset{t\to 0}{\longrightarrow} 0\,. \tag{1.4.10}$$

As a consequence of (1.4.8), (1.4.9), (1.4.10), we have

$$\frac{1}{t\mu(B_r(\bar{x}))}\Big(\int\limits_{B_r(\bar{x}+te_i)\setminus B_r(\bar{x})} u - \int\limits_{C_t} u\Big) \underset{t\to 0}{\longrightarrow} 0\,. \tag{1.4.11}$$

We used this device in order to integrate over C_t in place of over $B_r(\bar{x}+te_i)\setminus B_r(\bar{x})$, and we can evaluate the integral over C_t using Lemma 1.2.1. Namely, putting $\partial_\pm = \big\{(y,z) \in \partial B_r(\bar{x}) : \pm(z-\bar{z}) \ge 0\big\}$, let ∂_+ be represented as $\psi(\overline{A})$. Then, putting $\phi(y,\tau) = \psi(y)+\tau e_n$, we have $C_t = \phi(\overline{A}\times]0,t])$. Hence, by Lemma 1.2.1,

$$\int\limits_{C_t} u = \int\limits_0^t \Big(\int\limits_{\partial_+ + \tau e_n} u\nu_{\partial_+ + \tau} \cdot e_n\Big)\, d\tau\,.$$

Hence, differentiating and taking into account (1.4.11),

$$\lim_{t\to 0}\frac{1}{t}\int\limits_{B_r(\bar{x}+te_i)\setminus B_r(\bar{x})} u=\lim_{t\to 0}\frac{1}{t}\int\limits_{C_t} u=\int\limits_{\partial_+} u\nu_{\partial_+}\cdot e_n$$

By a similar argument,

$$\frac{1}{t}\int\limits_{B_r(\bar{x})\setminus B_r(\bar{x}+te_i)} u\xrightarrow[t\to 0]{}-\int\limits_{\partial_-} u\nu_{\partial_-}\cdot e_n$$

Hence, using (1.4.7),

$$\frac{u(\bar{x}+te_i)-u(\bar{x})}{t}\xrightarrow[t\to 0]{}\frac{1}{\mu(B_r(\bar{x}))}\int\limits_{\partial B_r(\bar{x})} u\nu_{\partial B_r(\bar{x})}\cdot e_n$$

Using the divergence Theorem for the function $(0,0,...,u)$, we have

$$\exists\,\frac{\partial u}{\partial x_i}(\bar{x})=\frac{1}{\mu(B_r(\bar{x}))}\int\limits_{\partial B_r(\bar{x})} u\nu_{\partial B_r(\bar{x})}\cdot e_i=\frac{1}{\mu(B_r(\bar{x}))}\int\limits_{\partial B_r(\bar{x})}\frac{\partial u}{\partial x_i}$$

thus $\frac{\partial u}{\partial x_i}$ exists on Ω and is continuous, and also has the mean value property. By induction, u is of class C^∞ (and the derivatives to u of any order have the mean value property), in particular of class C^2, and, by Theorem 1.1.2, u is harmonic. ∎

Remark 1.4.9. Note that the proof of Theorem 1.4.8 shows that if u is harmonic, then the derivatives of u of any order are also harmonic. ∎

1.5 Dirichlet Problem on the Ball

Lemma 1.4.1 shows that the value of a harmonic at the center of a ball only depends on the values on the boundary of the ball, and in fact, gives the value at the center in terms of the values on the boundary. In this section, we will try to do an analog for points in the ball different from its center, and also on open sets different from the ball. Note that Corollary 1.4.7 states that in any case, the values of a harmonic function on an open set only depend on its values on the boundary of the set, and the point is to explicitly find such a dependence. We will now use the functions $\bar{\psi}$, $\bar{\phi}$, defined in Section 1.3. Let E be so that $\overline{E}\subseteq\Omega$ and let $\bar{x}\in E$, and let $\bar{r}>0$ be so that $\overline{B_{\bar{r}}(\bar{x})}\subseteq E$. Suppose also E is a connected open set, div-regular. We mimic the proof of Lemma 1.4.1. We proceed in the same way, with E in place of $B_r(\bar{x})$, the unique difference being that, in the present case the function v defined by $v(x)=\bar{\phi}(x-\bar{x})$ is not necessarily constant on ∂E, so that the integral $\int\limits_{\partial E}(v\ \mathrm{grad}\, u)\cdot\nu$ does not vanish. We thus obtain

$$u(\bar{x})=\int\limits_{\partial E} u(x)\ \mathrm{grad}\,\bar{\phi}(x-\bar{x})\cdot\nu(x)\,dx-\int\limits_{\partial E}\bar{\phi}(x-\bar{x})\ \mathrm{grad}\,u(x)\cdot\nu(x)\,dx \tag{1.5.1}$$

On the other hand, by Corollary 1.3.5, if v is any harmonic function on all of E (that is, unlike $\bar\phi(x-\bar x)$, having no singularities in E), continuous on $\overline{E}$, we have

$$0=\int\limits_{\partial E} u(x)\ \mathrm{grad}\, v(x)\cdot\nu(x)\,dx-\int\limits_{\partial E} v(x)\ \mathrm{grad}\, u(x)\cdot\nu(x)\,dx\,.$$

Consequently, if

$$G(x,y)=\bar\phi(x-y)+h(x,y)\,, \tag{1.5.2}$$

with $h:\overline{E}\times E$, harmonic on E and continuous on $\overline{E}$, with respect to x, and such that

$$G(x,y)=0\quad \forall\, x\in\partial E\,, \tag{1.5.3}$$

summing the previous two identities, we get

$$u(\bar x)=\int\limits_{\partial E} u(x)\ \mathrm{grad}_x\, G(x,\bar x)\cdot\nu(x)\,dx\,, \tag{1.5.4}$$

where of course, grad_x denotes the gradient with respect to the variable x. Such a function G, if exists, is called *Green function on E*. The Green function, if exists, is unique. In fact, if G_1, G_2 are two functions having the previous properties, then, for given $y\in E$, the function $x\mapsto G_1(x,y)-G_2(x,y)$ is defined over all $\overline{E}$ (in fact the difference does not contain the singular term $\bar\phi(x-y)$), continuous on $\overline{E}$, and harmonic on E, and attains the value 0 on all of ∂E, so that, by the maximum principle is identically 0 on E. In conclusion, we have in fact represented u on E in terms of the values of u on ∂E, *provided we know a Green function on E*. Now, the point is to find a Green function. This is in general a nontrivial problem. We will now consider the case of a ball, that is, $E=B_R(0)$. In this specific case, we can find explicitly the Green function. Namely.

$$G(x,y)=\begin{cases}\bar\phi(x-y)-\bar\phi\Big(\frac{||y||}{R}(x-\bar y)\Big) & \text{if } y\neq 0\\ \bar\phi(x)-\bar\psi(R) & \text{if } y=0\end{cases}\qquad \bar y:=\frac{R^2y}{||y||^2}\,.$$

Note that $\bar y$ is a sort of inverse of y with respect to $B_R(0)$, in the sense that y and $\bar y$ are in the same half-line with endpoint 0, and $||y||\ ||\bar y||=R^2$. As $||\bar y||>R$ for $y\in B_R(0)$, the argument $\frac{||y||}{R}(x-\bar y)$ of $\bar\phi$ in the definition of G is different from 0, so that $\bar\phi$ is defined there. We will now prove that $\bar\phi(x-y)-\bar\phi\Big(\frac{||y||}{R}(x-\bar y)\Big)=0$ if $||x||=R$, so that (1.5.3) holds and G is in fact the Green function. It suffices to prove that, if $||x||=R$, then $||x-y||=\Big|\Big|\frac{||y||}{R}(x-\bar y)\Big|\Big|$. This fact has also a simple geometrical interpretation. We have

$$||x-y||^2=||x||^2+||y||^2-2x\cdot y=R^2+||y||^2-2x\cdot y\,,$$

$$||x-\bar y||^2=||x||^2+\frac{R^4}{||y||^2}-2x\cdot\frac{R^2y}{||y||^2}=R^2+\frac{R^4}{||y||^2}-\frac{2R^2}{||y||^2}x\cdot y\,,$$

thus, $||x-\bar{y}||^2 = \left(\frac{R}{||y||}\right)^2 ||x-y||^2$, and $||x-y|| = \left|\left|\frac{||y||}{R}(x-\bar{y})\right|\right|$. In conclusion, G has all properties required for a Green function. Note that the definition of $G(x,y)$ for $y=0$ is in some sense natural, in that, by definition $\bar{\phi}(x) = \bar{\psi}(||x||)$, and

$$\left|\left|\frac{||y||}{R}(x-\bar{y})\right|\right| \underset{y\to 0}{\longrightarrow} R\,.$$

Note also that, a simple calculation shows that

$$G(x,y) = \bar{\psi}\Big(\sqrt{||x||^2+||y||^2-2x\cdot y}\Big) - \bar{\psi}\left(\sqrt{\frac{1}{R^2}\big(||x||^2||y||^2+R^4-2R^2x\cdot y\big)}\;\right)$$

when $y \neq 0$, but also when $y=0$. This proves that G is symmetric (that is, $G(x,y) = G(y,x)$ whenever $x \in \overline{B_R(0)}$, $y \in B_R(0)$, $x \neq y$). In order to give an explicit representation of the harmonic function u in terms of its values on the boundary via (1.5.4) in this specific case, we need to give a formula for $\operatorname{grad}_x G(x,\bar{x}) \cdot \nu(x)$. It is possible to verify that

$$\operatorname{grad}_x G(x,y)\cdot\nu(x) = \frac{R^2-||y||^2}{\bar{d}_n R}\,\frac{1}{||x-y||^n}\,. \tag{1.5.5}$$

Hence,

$$u(y) = \frac{R^2-||y||^2}{\bar{d}_n R}\int\limits_{\partial B_R(0)} u(x)\,\frac{1}{||x-y||^n}\,dx\,. \tag{1.5.6}$$

Note that, in some sense, formula (1.5.6) is an analog of the Cauchy formula for functions of one complex variable. Putting $u=1$ in (1.5.6), we get

$$1 = \frac{R^2-||y||^2}{\bar{d}_n R}\int\limits_{\partial B_R(0)} \frac{1}{||x-y||^n}\,dx\,. \tag{1.5.7}$$

We will now consider a sort of an inverse problem, that is, given a continuous function f on $\partial B_R(0)$, putting

$$u(y) = \begin{cases} \dfrac{R^2-||y||^2}{\bar{d}_n R}\displaystyle\int\limits_{\partial B_R(0)} f(x)\,\frac{1}{||x-y||^n}\,dx & \text{if } y \in B_R(0) \\ f(y) & \text{if } y \in \partial B_R(0) \end{cases} \tag{1.5.8}$$

can we state that u is continuous on $\overline{B_R(0)}$ and harmonic on $B_R(0)$? The answer to this question is provided by the following theorem.

Theorem 1.5.1. *Given $f : \partial B_R(0) \to \mathbb{R}$, continuous, then the function u defined by (1.5.8) is continuous on $\overline{B_R(0)}$ and harmonic on $B_R(0)$.*

Proof. We use the equivalent formula (see (1.5.5))

$$u(y) = \int_{\partial B_R(0)} f(x) \mathrm{grad}_x G(x,y) \cdot \nu(x)\, dx$$

valid for $y \in B_R(0)$. It is easy to see that we can differentiate under integral, taking into account that for y' in a neighbourhood of y and $x \in \partial B_R(0)$, $||x - y'|| \geq d > 0$, so that $\mathrm{grad}_x G(x,y)$ is bounded (cf. Lemma 1.3.2 i). Thus

$$\begin{aligned}\Delta u(y) &= \int_{\partial B_R(0)} f(x) \Delta_y \Big(\sum_{i=1}^n \frac{\partial G(x,y)}{\partial x_i} \nu_i(x) \Big) dx \\ &= \int_{\partial B_R(0)} f(x) \sum_{i=1}^n \frac{\partial}{\partial x_i} \big(\Delta_y G(x,y) \big) \nu_i(x)\, dx = 0\end{aligned}$$

where in the second equality we have exchanged the differentiation operators, and this is possible by the Schwarz Theorem, and in the third we have used that G is harmonic in x and symmetric, thus harmonic in y. Therefore, u is harmonic on $B_R(0)$. We now prove that u is continuous on $\overline{B_R(0)}$. This is obvious on $B_R(0)$, and, since the restriction to $\partial B_R(0)$ is continuous by hypothesis, it suffices to prove

$$u|_{B_R(0)}(y) \underset{y \to \bar{y}}{\longrightarrow} f(\bar{y}) \quad \forall \bar{y} \in \partial B_R(0)\,. \tag{1.5.9}$$

For simplicity put $P(x,y) = \dfrac{R^2 - ||y||^2}{\bar{d}_n R} \dfrac{1}{||x-y||^n}$, where P stands for Poisson (in fact P is called *Poisson Kernel*), so that u assumes the form

$$u(y) = \int_{\partial B_R(0)} f(x) P(x,y) dx\,. \tag{1.5.10}$$

for $y \in B_R(0)$. Using (1.5.7), we get

$$u(y) - f(\bar{y}) = \int_{\partial B_R(0)} (f(x) - f(\bar{y})) P(x,y) dx\,. \tag{1.5.11}$$

Given $\varepsilon > 0$, let $\delta > 0$ be so that,

$$x \in \partial B_R(0) \cap B_\delta(\bar{y}) \Rightarrow |f(x) - f(\bar{y})| < \varepsilon\,.$$

Then,

$$\begin{aligned}&\Big| \int_{\partial B_R(0) \cap B_\delta(\bar{y})} (f(x) - f(\bar{y})) P(x,y) dx \Big| \\ &\qquad \leq \varepsilon \int_{\partial B_R(0) \cap B_\delta(\bar{y})} P(x,y) dx\end{aligned}$$

$$\leq \varepsilon \int_{\partial B_R(0)} P(x,y)dx = \varepsilon\,,$$

where in the last inequality, we used the fact that, by definition, $P(x,y) > 0$. On the other hand, suppose now $||y - \bar{y}|| < \frac{\delta}{2}$, so that, if $x \in \partial B_R(0) \setminus B_\delta(\bar{y})$, we have $||x - y|| \geq \frac{\delta}{2}$, hence, by the definition of P,

$$\Big| \int_{\partial B_R(0)\setminus B_\delta(\bar{y})} (f(x) - f(\bar{y}))P(x,y)dx\Big|$$

$$\leq \Big(\frac{2}{\delta}\Big)^n \frac{R^2 - ||y||^2}{\bar{d}_n R} 2M \int_{\partial B_R(0)} 1 \underset{y\to\bar{y}}{\longrightarrow} 0$$

where $M := \max|f|$, as $||\bar{y}||^2 = R^2$. By the previous inequalities we have

$$\Big| \int_{\partial B_R(0)} (f(x) - f(\bar{y}))P(x,y)dx\Big| < 2\varepsilon$$

for $||y - \bar{y}||$ sufficiently small, and, in view of (1.5.11), (1.5.9) is proved. ∎

In the following, we will denote by

$$H_B(f)$$

the continuous function on the closure of the ball B, harmonic on B, that amounts to the continuous function f on ∂B, as given by (1.5.8). The problem of finding a harmonic function on Ω with prescribed (continuous) values on its boundary is called *Dirichlet problem*. Thus, Theorem 1.5.1 states that the function u given by (1.5.8) solves the Dirichlet problem on the ball $B_R(0)$ with values f on the boundary. We will now see some consequences of Theorem 1.5.1 and related results. First, we give another (and simpler) proof of Theorem 1.4.8.

Given u as in Theorem 1.4.8, and $B := B_R(\bar{x})$ such that $\overline{B_R(\bar{x})} \subseteq \Omega$, let $v = H_B(u)$. Then, $u - v$ has the mean value property on Ω, and is identically 0 on $\partial B_R(\bar{x})$, hence by the maximum principle (cf. Remark 1.4.6), it is identically 0 on $\overline{B_R(\bar{x})}$. Thus u amounts to v, hence is harmonic on $B_R(\bar{x})$. Since B is an arbitrary ball, whose closure is contained in Ω, u is harmonic on Ω. ∎

Remark 1.5.2. The previous proof shows that, in order that u be harmonic, it suffices that the mean property holds locally in the sense of Remark 1.4.6. ∎

Corollary 1.5.3. *If u_h is a sequence of harmonic functions on Ω, such that $u_h \underset{h\to+\infty}{\longrightarrow} u$, uniformly on compact subsets of Ω, then u is harmonic on Ω.*

Proof. Let $\bar{x} \in \Omega$ and let $r > 0$ be such that $\overline{B_r(\bar{x})} \subseteq \Omega$. Since every u_h satisfies (1.4.5), taking the limit, as $u_h \underset{h\to+\infty}{\longrightarrow} u$ uniformly on $B_r(\bar{x})$, then u satisfies (1.4.5) as well. By Theorem 1.4.8 u is harmonic. ∎

Theorem 1.5.4 (Harnack inequality. *Let F be a nonempty connected and compact subset of Ω. Then, there exists $c > 0$ such that for every nonnegative harmonic function u on Ω, then $\sup_F u \le c \inf_F u$.*

Proof. Suppose first F has the form $F = \overline{B_r(\bar{x})}$, with $\bar{x} \in \Omega$, and r so small that

$$\overline{B_{4r}(\bar{x})} \subseteq \Omega\,. \tag{1.5.12}$$

If $x_1, x_2 \in F$, by the definition of F (and the triangular inequality), we have $||x_1 - x_2|| \le 2r$. Therefore, also using (1.5.12),

$$B_r(x_1) \subseteq B_{3r}(x_2) \subseteq B_{4r}(\bar{x}) \subseteq \Omega\,. \tag{1.5.13}$$

Now, we have $\mu(B_r(x)) = \omega_n r^n$, for every $x \in \mathbb{R}^n$ and for a suitable $\omega_n > 0$. Therefore, as u is assumed to be nonnegative, and in view of the mean value Theorem, we have

$$u(x_1) = \frac{1}{\omega_n r^n} \int\limits_{B_r(x_1)} u \le \frac{1}{\omega_n r^n} \int\limits_{B_{3r}(x_2)} u = 3^n \frac{1}{\omega_n (3r)^n} \int\limits_{B_{3r}(x_2)} u = 3^n u(x_2)\,. \tag{1.5.14}$$

Thus, we have prove the Theorem for F of the previous form with $c = 3^n$. In the general case, by a compactness argument, F is covered by finitely many balls $F_i = B_{r_i}(x_i)$, $i = 1, ..., N$, satisfying (1.5.12), and we can assume $F_i \cap F \neq \emptyset$ for each $i = 1, ..., N$. We will denote by $\mathcal{F}$ the set of all F_i. We will say that F_i is 1-connected to $F_{i'}$ if $F_i \cap F_{i'} \neq \emptyset$, and by induction that F_i and $F_{i'}$ are $(h+1)$- connected if F_i is h-connected to some F_j that is 1-connected to $F_{i'}$. Finally, we say that F_i is connected to $F_{i'}$ if it is h-connected to $F_{i'}$ for some h. We have that F_i is connected to any $F_{i'}$. In fact, in the opposite case, putting V to be the union of F_i connected to F_1, and V' to be the union of F_i not connected to F_1, we split F into the union of two nonempty sets, open in F, in the following way

$$F = \big(F \cap V\big) \cup \big(F \cap V'\big), \quad \big(F \cap V\big) \cap \big(F \cap V'\big) = \emptyset$$

and this contradicts the connectedness of F. Using (1.5.14) and a recursive argument, we get that if $x_1 \in F_{i_1}$ and $x_2 \in F_{i_2}$, and F_{i_1} and F_{i_2} are h-connected, then $u(x_1) \le c^{(h+1)n} u(x_2)$. Since any two elements of $\mathcal{F}$ are connected, in fact (as every sequence of different elements of $\mathcal{F}$ contains at most N elements), $(N-1)$- connected, we have

$$u(x_1) \le c^{Nn} u(x_2)$$

and the Theorem is proved with $c = Nn$. ∎

Remark 1.5.5. The important point in the previous theorem is that we can choose the constant c *independent of the function u.* The constant, instead, depends on the subset F.

The theorem would be false if we would replace F with all Ω. Moreover, note that the 1-dimensional version of Theorem 1.5.4 is trivial. In fact, in this case the harmonic functions are affine. ■

Corollary 1.5.6. *If u is harmonic on Ω, and $\bar{x} \in \Omega$, and $\overline{B_r(\bar{x})} \subseteq \Omega$, then $||grad\, u(\bar{x})|| \leq \max\limits_{\partial B_r(\bar{x})} |u| \frac{n}{r}$.*

Proof. We have (see last formula in proof of Theorem 1.4.8)

$$\frac{\partial u}{\partial x_i}(\bar{x}) = \frac{1}{\mu(B_r(\bar{x}))} \int\limits_{\partial B_r(\bar{x})} u\nu \cdot e_i$$

Hence,

$$\mathrm{grad} u(\bar{x}) = \frac{1}{\mu(B_r(\bar{x}))} \int\limits_{\partial B_r(\bar{x})} u\nu \,,$$

$$||\mathrm{grad} u(\bar{x})|| \leq \frac{1}{\mu(B_r(\bar{x}))} \int\limits_{\partial B_r(\bar{x})} |u| \leq \max_{\partial B_r(\bar{x})} |u| \frac{n}{r} \,,$$

the last inequality being a consequence of Corollary 1.2.3. ■

Corollary 1.5.7. (Liouville Theorem). *A bounded harmonic function u on $\mathbb{R}^n$ is constant.*

Proof. Fix $x \in \mathbb{R}^n$. For any $r > 0$ we have $\overline{B_r(x)} \subseteq \mathbb{R}^n$, so that, putting $M = \sup |u|$, we have $M < +\infty$ by the hypothesis, and by Corollary 1.5.6, $||\mathrm{grad} u(x)|| \leq M\frac{n}{r}$. Since this inequality holds for every $r > 0$, we have $\mathrm{grad} u(x) = 0$ for every $x \in \mathbb{R}^n$, hence u is constant. ■

Note that the Liouville Theorem has an analog in the theory of one complex variable. The following theorem is a sort of Ascoli-Arzelà Theorem for harmonic functions. Note that, unlike the general case, it suffices the harmonic functions are equibounded.

Theorem 1.5.8. *Every sequence u_h of harmonic functions on Ω, equibounded on every compact subset of Ω, has a subsequence uniformly convergent on every compact subset of Ω to a harmonic function on Ω.*

Proof. It is a standard fact that Ω is the union of countably many compact subsets. Namely,

$$\Omega = \bigcup_{m=1}^{\infty} \Omega_m, \quad \Omega_m =: \left\{ x \in \mathbb{R}^n : ||x|| \leq m, d(x, \Omega^c) \geq \frac{1}{m} \right\}.$$

where, in the case $\Omega = \mathbb{R}^n$ we use the convention $d(x, \emptyset) = +\infty$. More precisely, it is easy to verify that every compact subset of Ω is contained in some Ω_m. Thus it suffices to prove that, for every m, u_h has a subsequence uniformly convergent on Ω_m. In fact, by a diagonal argument, we then obtain a subsequence uniformly convergent on every Ω_m. Given a compact subset of Ω, as it is contained in some Ω_m, such a subsequence is uniformly convergent on K. Given $x \in \Omega_m$, we have $\overline{B_{\frac{1}{2m}}(x)} \subseteq \Omega_{2m}$. As u_h are equibounded, say $|u_h| \leq M$ on Ω_{2m}, in view of Corollary 1.5.6 we have

$$||\mathrm{grad} u_h(y)|| \leq \max_{\partial B_{\frac{1}{4m}}(y)} |u_h| 4nm \leq 4nmM$$

for every $y \in B_{\frac{1}{4m}}(x)$, as in such a case we have

$$\overline{B_{\frac{1}{4m}}(y)} \subseteq \overline{B_{\frac{1}{2m}}(x)} \subseteq \Omega_{2m}$$

Therefore,

$$|u_h(x_1) - u_h(x_2)| = |\mathrm{grad} u_h(y) \cdot (x_1 - x_2)| \leq 4nmM||x_1 - x_2||$$

for some y in the segment-line with endpoints x_1 and x_2, for every $x_1, x_2 \in \overline{B_{\frac{1}{4m}}(x)}$. It follows that the sequence u_h is equibounded and equicontinuous on $\overline{B_{\frac{1}{4m}}(x)}$, thus it has a subsequence uniformly convergent on $\overline{B_{\frac{1}{4m}}(x)}$, a fortiori on $B_{\frac{1}{4m}}(x)$. As, by a compactness argument, Ω_m is covered by finitely many balls of the form $B_{\frac{1}{4m}}(x)$ with $x \in \Omega_m$, u_h has a subsequence uniformly convergent on Ω_m. The limit function is harmonic by Corollary 1.5.3. ∎

1.6 Dirichlet Problem on General Domains

In this section, we will solve the Dirichlet problem for general domains. However, we will have to require some relatively mild conditions on the domain. We will now introduce a new class of functions, the subharmonic functions, which are a variant of the harmonic function in the sense that they take at a point a value which is smaller than or equal to the average on a ball centered at the point. In the one-dimensional case, as previously seen, the harmonic functions are the affine ones. Instead, the subharmonic functions are the convex ones.

Definition 1.6.1. *A continuous function from Ω to $\mathbb{R}$ is said to be subharmonic if, for every $x \in \Omega$ and for every $r > 0$ such that $\overline{B_r(x)} \subseteq \Omega$, we have*

$$u(x) \leq \frac{1}{\mu(B_r(x))} \int_{B_r(x)} u \,. \tag{1.6.1}$$

Lemma 1.6.2 (strong maximum principle). *If Ω is bounded and $u : \overline{\Omega} \to \mathbb{R}$ is continuous and subharmonic on Ω, then $\max\limits_{\overline{\Omega}} u = \max\limits_{\partial\Omega} u$ and if u attains a maximum value on Ω, then it is constant on $\overline{\Omega}$.*

Proof. Suppose $\bar{x} \in \Omega$ and $u(\bar{x}) = \max_{\Omega} u$. Then, take $r > 0$ so that $\overline{B_r(x)} \subseteq \Omega$. By Definition 1.6.1,

$$\int_{B_r(x)} (u - u(\bar{x})) \geq 0$$

and we proceed in the exactly same way as in proof of Theorem 1.4.4 (and Corollary 1.4.5). ∎

Remark 1.6.3. Note that for subharmonic functions an analogous minimum principle does not hold. ∎

Lemma 1.6.4. *Given a continuous function $u : \Omega \to \mathbb{R}$, the following are equivalent:*
i) u is subharmonic.
ii) For every $x \in \Omega$ there exists $\bar{r} > 0$ such that (1.6.1) holds for every $r \in]0, \bar{r}[$.
iii) For every $\bar{x} \in \Omega$ and every $r > 0$ such that $\overline{B_r(\bar{x})} \subseteq \Omega$, if h is a continuous function on $\overline{B_r(\bar{x})}$, harmonic on $B_r(\bar{x})$ and $h \geq u$ on $\partial B_r(\bar{x})$, then $h \geq u$ on $\overline{B_r(\bar{x})}$.

Proof. Clearly, i)$\Rightarrow$ ii). Prove ii)$\Rightarrow$iii). If ii) holds, given h as in iii), by (1.6.1) and the mean value property, $u - h$ satisfies ii) as well. Thus, $u - h$ satisfies the maximum principle, as the proof of Lemma 1.6.2 in fact, only uses ii). As $u - h \leq 0$ on $\partial B_r(\bar{x})$, and $u - h$ attains its maximum on $\overline{B_r(\bar{x})}$ at points in $\partial B_r(\bar{x})$, we thus have $u - h \leq 0$ on $\overline{B_r(\bar{x})}$, and iii) holds. Prove now iii)$\Rightarrow$ i). Let $r > 0$ be such that $\overline{B_r(\bar{x})} \subseteq \Omega$. Let $h := H_{B_r(\bar{x})}(u)$ so that, by definition, $h = u$ on $\partial B_r(\bar{x})$, hence by iii), $h \geq u$ on $B_r(\bar{x})$. Then, by Lemma 1.4.1, we have

$$u(\bar{x}) \int_{\partial B_t(\bar{x})} 1 \leq h(\bar{x}) \int_{\partial B_t(\bar{x})} 1 = \int_{\partial B_t \bar{x})} h = \int_{\partial B_t \bar{x})} u$$

for every $t \in]0, r[$. Thus, integrating with respect to $t \in]0, r[$, we get

$$\mu(B_r(\bar{x}))u(\bar{x}) = u(\bar{x}) \int_0^r \Big(\int_{\partial B_t(\bar{x})} 1 \Big) \, dt \leq \int_0^r \Big(\int_{\partial B_t \bar{x})} u \Big) \, dt = \int_{B_r(\bar{x})} u$$

and u is subharmonic. ∎

The harmonic functions are by definition the functions u of class C^2 such that $\Delta u = 0$. A similar characterization holds for the subharmonic functions. Note however, that in the present case the statement is slightly different from that for harmonic functions, that is, we characterize only the subharmonic functions of class C^2, not all subharmonic functions.

Lemma 1.6.5. *A function u of class C^2 on Ω is subharmonic if and only if $\Delta u \geq 0$ on Ω.*

Proof. If u is subharmonic, then $\Delta(u) \geq 0$ on Ω by Theorem 1.1.2. Suppose now Δu is *strictly* positive on Ω. Then, u is subharmonic by Theorem 1.1.2 again and Lemma 1.6.4 (more precisely, condition ii) there.) Suppose now $\Delta u \geq 0$ on Ω, and put $u_t(x) =$

$u(x)+t||x||^2$ for $t>0$. We have $\Delta u_t=\Delta u+2tn>0$. Thus, u_t is subharmonic by the previous considerations, and by passing to the limit as $t\to 0$ in (1.6.1), we get that u is subharmonic. ■

Lemma 1.6.6. *If u_i, $i=1,..h$ are subharmonic functions on Ω, then $u:=\max\limits_{i=1,\ldots,h} u_i$ is subharmonic on Ω.*

Proof. If $x\in\Omega$, and $\overline{B_r(x)}\subseteq\Omega$, we have $u(x)=u_i(x)$ for some $i=1,...,h$, thus

$$u(x)=u_i(x)\le\frac{1}{\mu(B_r(x))}\int\limits_{B_r(x)} u_i\le\frac{1}{\mu(B_r(x))}\int\limits_{B_r(x)} u\,. \qquad ■$$

Given a ball B whose closure in contained in Ω, and u subharmonic on Ω, we define the function $\overline{H}_B(u)$ (the harmonic lifting of u on B) as

$$\overline{H}_B(u)(x)=\begin{cases} H_B(u)(x) & \text{if } x\in B\\ u(x) & \text{if } x\notin B\end{cases}.$$

Note that, by Lemma 1.6.4 and the definition of $H_B(u)$, we have $\overline{H}_B(u)\ge u$ on Ω.

Lemma 1.6.7. *The function $\overline{H}_B(u)$ is subharmonic on Ω.*

Proof. We use ii) of Lemma 1.6.4. As the restrictions of u to B and to $\Omega\setminus\overline{B}$ are subharmonic, ii) clearly holds when $x\notin\partial B$. If $x\in\partial B$, then, for every $r>0$ such that $\overline{B_r(x)}\subseteq\Omega$ we have

$$\overline{H}_B(u)(x)=u(x)\le\frac{1}{\mu(B_r(x))}\int\limits_{B_r(x)} u\le\frac{1}{\mu(B_r(x))}\int\limits_{B_r(x)}\overline{H}_B(u)\,. \qquad ■$$

We now show the Perron method to solve the Dirichlet problem on a region Ω. Let Ω be bounded and let f be a continuous function on $\partial\Omega$. We define the space of *subsolutions* of the Dirichlet problem, namely let

$$S_f=\left\{u:\overline{\Omega}\to\mathbb{R}, u \text{ continuous on } \overline{\Omega},\ \text{subharmonic on } \Omega: u\le f \text{ on } \partial\Omega\right\}.$$

Theorem 1.6.8. *The function w defined by*

$$w=\sup\left\{u:u\in S_f\right\}$$

is harmonic on Ω.

Proof. If $u\in S_f$, as $u\le\max f$ on $\partial\Omega$, then $u\le\max f$ on Ω, by the maximum principle. As a consequence, $w\le\max f$. Also, note that, if B is a ball whose closure is contained in Ω, and $u\in S_f$, then

$$\max\{u, \min f\}, \overline{H}_B(u) \in S_f\,.$$

Fix $x_0 \in \Omega$, and we will prove that w is harmonic on a suitable ball centered at x_0. As a consequence, w will turn out to be harmonic on all of Ω. We will denote by B a ball centered at x_0 whose closure is contained in Ω. Let u_h be a sequence of function in S_f such that

$$u_h(x_0) \underset{h\to+\infty}{\longrightarrow} w(x_0)\,, \tag{1.6.2}$$

We can assume $u_h \geq \min f$, as in any case we can replace u_h by $\max\{u_h, \min f\} \in S_f$. Let $V_h = \overline{H}_B(u_h) \in S_f$. As V_h are harmonic functions on B, and $\min f \leq V_h \leq \max f$, by Theorem 1.5.8 there exist a subsequence V_{h_k} of V_h and a harmonic function V on B such that $V_{h_k} \underset{k\to+\infty}{\longrightarrow} V$, uniformly on the compact subsets of B. As $u_{h_k} \leq V_{h_k} \leq w$, in view of (1.6.2), we have

$$V(x) \leq w(x) \quad \forall\, x \in B, \qquad V(x_0) = w(x_0)\,. \tag{1.6.3}$$

As V is harmonic on B, it suffices to prove $V(x) = w(x)$ for all $x \in B$. By contradiction, suppose $\exists\, x_1 \in B : V(x_1) < w(x_1)$. Then,

$$\exists v \in S_f : v(x_1) > V(x_1)\,. \tag{1.6.4}$$

We set $d_h = \max\{V_h, v\}$, and $D_h = \overline{H}_B d_h$. As $\min_{\overline{\Omega}} v \leq v \leq D_h \leq w \leq \max f$ on B, D_{h_k} is a sequence of functions, equibounded and harmonic on B, so that it has a subsequence, uniformly convergent to a harmonic function D on the compact subsets of B. As $D_h \geq V_h$ by definition, we have

$$V \leq D \leq w \tag{1.6.5}$$

on B. On the other hand, by (1.6.3), $D(x_0) = V(x_0)$, and by (1.6.5), the function $D - V$, harmonic on B, attains the minimum value 0 on B at x_0, so that by the maximum principle we have $D = V$ on B. But, in contrast, $D(x_1) \geq v(x_1) > V(x_1)$, a contradiction. ∎

Our aim is to solve the Dirichlet problem. By this point of view, Theorem 1.6.8 does not suffice as, in general, the equality $w = f$ on $\partial\Omega$ is not necessarily valid. The validity of such an equality depends on some specific properties of the boundary. To clarify this point, we need the notion of *barrier*. We say that a continuous function u on Ω is *superharmonic* if, for every $x \in \Omega$ and for every $r > 0$ such that $\overline{B_r(x)} \subseteq \Omega$, we have $u(x) \geq \frac{1}{\mu(B_r(x))} \int\limits_{B_r(x)} u$, in other words if $-u$ is subharmonic. The previous results on subharmonic functions can be easily converted into results on superharmonic functions.

Definition 1.6.9. *A barrier at the point $\xi \in \partial\Omega$ is a continuous function $\overline{b}$ on $\overline{\Omega}$, superharmonic on Ω, such that $\overline{b} \geq 0$ on $\overline{\Omega}$, and $\overline{b}(x) = 0 \iff x = \xi$.*

A case where we have the existence of a barrier at a point $\xi \in \partial\Omega$ is when the *exterior sphere condition* is satisfied, i.e.,

there exists a ball $B := B_R(\bar{x})$ *such that* $\overline{B} \cap \overline{\Omega} = \{\xi\}$.

In such a case, a barrier is given by the function $\bar{b}(x) = \overline{\phi}(x - \bar{x}) - \overline{\phi}(\xi - \bar{x})$, where the function $\overline{\phi}$ is defined in Section 3. In fact, definitely $\bar{x} \notin \overline{\Omega}$, as, in the opposite case, $B \cap \Omega$ would contain a point, hence a nonempty open set in $\mathbb{R}^n$. Therefore, $\overline{w}$ is defined and continuous on $\overline{\Omega}$, and harmonic on Ω. Moreover, if $x \in \overline{\Omega} \setminus \{\xi\}$, then $x \notin \overline{B}$, hence $||x - \bar{x}|| > R = ||\xi - \bar{x}||$, and as $\overline{\psi}$ is strictly increasing, then $\bar{b}(x) > 0$. As an important and known particular case, the exterior sphere condition is satisfied at all points of $\partial\Omega$ when $\partial\Omega$ is of class C^2.

Theorem 1.6.10. *If there exists a barrier at any point of $\partial\Omega$, then the function $\tilde{w}$ defined by*

$$\tilde{w}(x) = \begin{cases} w(x) & \text{if } x \in \Omega \\ f(x) & \text{if } x \in \partial\Omega \end{cases}$$

where w is the function defined in Theorem 1.6.8, is continuous on $\overline{\Omega}$, harmonic on Ω, hence solves the Dirichlet problem with values f on $\partial\Omega$.

Proof. In view of Theorem 1.6.8, we have only to prove that, for every $\xi \in \partial\Omega$, we have

$$\tilde{w}|_{\Omega}(x) \underset{x \to \xi}{\longrightarrow} \tilde{w}(\xi)\,. \tag{1.6.6}$$

Let $\xi \in \partial\Omega$, and let $\bar{b}$ be a barrier at ξ. Then, we have

$$\forall\, \varepsilon > 0 \;\exists\, k > 0 : \; |f(\xi) - f(x)| \le \varepsilon + k\bar{b}(x) \;\; \forall\, x \in \partial\Omega\,. \tag{1.6.7}$$

In fact, by continuity, there exists $\delta > 0$ such that every $x \in \partial\Omega \cap B_\delta(\xi)$ satisfies $|f(\xi) - f(x)| < \varepsilon$, and a fortiori, satisfies (1.6.7) for any positive k. On the other hand, as by definition $\bar{b} > 0$ on $\overline{\Omega} \setminus \{\xi\}$, by a compactness argument there exists $\eta > 0$ such that $\bar{b} \ge \eta$ on $\overline{\Omega} \setminus B_\delta(\xi)$, and taking k such that $k\eta > 2\max|f|$, also every $x \in \partial\Omega \setminus B_\delta(\xi)$ satisfies (1.6.7). By (1.6.7) the function $f(\xi) - \varepsilon - k\bar{b}$, subharmonic on Ω, is an element of S_f, thus

$$f(\xi) - \varepsilon - k\bar{b}(x) \le w(x) \;\; \forall\, x \in \Omega\,. \tag{1.6.8}$$

On the other hand, for every $u \in S_f$ we have, in view of (1.6.7) again, $u(x) \le f(x) \le f(\xi) + \varepsilon + k\bar{b}(x)$, thus $u(x) - \big(f(\xi) + \varepsilon + k\bar{b}(x)\big) \le 0$ for all $x \in \partial\Omega$, and, as the function

$$u - \big(f(\xi) + \varepsilon + k\bar{b}\big)$$

is subharmonic, by the maximum principle, $u - \big(f(\xi) + \varepsilon + k\bar{b}\big) \le 0$ on Ω. Thus, $u(x) \le f(\xi) + \varepsilon + k\bar{b}(x)$ for all $x \in \Omega$, for all $u \in S_f$. By the definition of w we thus have

$$w(x) \le f(\xi) + \varepsilon + k\bar{b}(x) \;\; \forall\, x \in \Omega\,. \tag{1.6.9}$$

Since $\bar{b}$ is continuous and $\bar{b}(\xi) = 0$, there exists $\delta > 0$ such that for all $x \in B_\delta(\xi)$ we have $\bar{b}(x) = |\bar{b}(x)| < \frac{\varepsilon}{k}$, hence, in view of (1.6.8) and (1.6.9),

$$|w(x) - f(\xi)| \le \varepsilon + k\bar{b}(x) < 2\varepsilon$$

and (1.6.6) immediately follows. ■

1.7 Other Boundary Conditions

We now wish to study the problem

$$\begin{cases} \Delta u = g & \text{on } \Omega \\ u = f & \text{on } \partial\Omega \end{cases} \qquad P_{f,g}$$

with f continuous on $\partial\Omega$, g continuous and bounded on Ω, and suppose Ω bounded. By solution of $P_{f,g}$ we mean a continuous function u on $\overline{\Omega}$, of class C^2 on Ω, that satisfies requirements of $P_{f,g}$. The first remark is that the solution to $P_{f,g}$, if exists, is unique. Indeed, if u_1 and u_2 are solutions of $P_{f,g}$, then $\bar{u} := u_1 - u_2$ is harmonic and satisfies $\bar{u} = 0$ on $\partial\Omega$, thus, by the maximum principle, $\bar{u} = 0$ on Ω. We will now study the problem of the existence of the solution of $P_{f,g}$. In general, we cannot obtain existence results with the above conditions on f and g even if, for example, Ω is a ball. In fact there are cases where $P_{f,g}$ has no solutions. We need some preliminary considerations. Note the following trivial fact, which will be useful in the following.

$$\frac{\partial\alpha(\overline{x} - \overline{y})}{\partial x_i} = -\frac{\partial\alpha(\overline{x} - \overline{y})}{\partial y_i} \tag{1.7.1}$$

if α is a function defined in an open set E in $\mathbb{R}^n$, and there exists $\dfrac{\partial\overline{\alpha}(\overline{z})}{\partial z_i}$, with $\bar{x} - \bar{y} = \bar{z} \in E$.

Lemma 1.7.1. *The integral*

$$\int\limits_{B_r(\bar{x})} ||x - \bar{x}||^\alpha \, dx$$

is finite if and only if $\alpha > -n$.

Proof. We have

$$\int\limits_{B_r(\bar{x})} ||x - \bar{x}||^\alpha \, dx = \int\limits_0^r \Big(\int\limits_{\partial B_t(\bar{x})} ||x - \bar{x}||^\alpha \, dx \Big) dt = \int\limits_0^r t^\alpha \frac{1}{\bar{d}_n} t^{n-1} \, dt = \frac{1}{\bar{d}_n} \int\limits_0^r t^{\alpha+n-1} \, dt$$

where $\bar{d}_n$ is defined in Section 1.3. Note that, to be precise, we should interpret the integrals in the previous formula as limits for $\varepsilon \to 0$ of integrals on $[\varepsilon, r]$ as we are using Corollary 1.2.2, valid when the integrand is continuous, and the function $||x||^\alpha$ is not continuous at 0 for negative α. We conclude recalling that the integral $\int\limits_0^r t^\beta \, dt$ is finite when $\beta > -1$. ■

We will now estimate the derivatives of $\overline{\phi}$. We have for some positive constants $c_{1,n}$, $c_{2,n}$, $\dfrac{\partial\overline{\phi}(x-y)}{\partial x_i} = c_{1,n}||x-y||^{1-n}\dfrac{(x-y)_i}{||x-y||}$, hence

$$\Big|\frac{\partial\overline{\phi}(x-y)}{\partial x_i}\Big| \le c_{1,n}||x-y||^{1-n} \tag{1.7.2}$$

and, by similar considerations,

$$\Big|\frac{\partial^2\overline{\phi}(x-y)}{\partial x_i \partial x_j}\Big| \le c_{2,n}||x-y||^{-n} \tag{1.7.3}$$

when $x \neq y$. Of course, when $x = y$, (1.7.2) and (1.7.3) do not make sense.

Corollary 1.7.2. *The functions $y \mapsto \overline{\phi}(y)$, $y \mapsto \dfrac{\overline{\phi}(y)}{||y||}$ and $y \mapsto \dfrac{\partial\overline{\phi}(y)}{\partial y_i}$ are summable on any ball with center at 0. The integral of such functions on $B_r(0)$ tends to 0 as $r \to 0$.*

Proof. The first statement easily follows from the definition of $\overline{\phi}$, (1.7.2), and Lemma 1.7.1. Note that in the case $n = 2$ we simply have to recall that the order of infinity, as $t \to 0^+$, of $\ln t$, is smaller than that of $t^{-\alpha}$ for any positive α, for example $t^{-\frac{1}{2}}$. The second statement could be directly verified. However, it also follows from the general result that the integral of a summable function on a set, tends to 0 as the measure of the set tends to 0 (absolute continuity of the integral of a summable function). ■

Note that the same considerations as before, in view of Lemma 1.7.1 and (1.7.3), do not permit to prove an analogous statement for the second derivatives, which in fact, is in general false. We now define the *Newtonian potential of g on Ω* as the function u defined by

$$u(x) = \int_\Omega \overline{\phi}(x-y)g(y)\,dy\,. \tag{1.7.4}$$

In some sense, this is a form of convolution of $\overline{\phi}$ and g. Note that u is defined on all of $\mathbb{R}^n$, as for every $x \in \mathbb{R}^n$ the integral in (1.7.4) is finite. In fact, the only singular point is $y = x$. Thus, it suffices to prove that the integral on a small ball with center at x is finite. This follows from the boundedness of g and Corollary 1.7.2.

Theorem 1.7.3. *If u is defined as in (1.7.4), then u is continuous. Moreover,*

$$\frac{\partial u}{\partial x_i}(x) = \int_\Omega \frac{\partial\overline{\phi}}{\partial x_i}(x-y)g(y)\,dy \quad \forall\, x \in \mathbb{R}^n\,. \tag{1.7.5}$$

Proof. Note that this theorem states that we can differentiate in (1.7.2) under integral sign. This is not trivial as the integrand function is singular. We introduce a function $\eta : \mathbb{R} \to \mathbb{R}$ of class C^∞ such that

$$0 \leq \eta \leq 1 \tag{1.7.6}$$

$$\eta = 0 \text{ on }]-\infty, 1], \quad \eta = 1 \text{ on } [2, +\infty[\tag{1.7.7}$$

$$0 \leq \eta' \leq 2 \tag{1.7.8}$$

Such a function can be easily constructed. Then, we put

$$\eta_\varepsilon(x,y) = \eta\Big(\frac{||x-y||}{\varepsilon}\Big),$$

$$u_\varepsilon(x) = \int\limits_\Omega \overline{\phi}(x-y)\eta_\varepsilon(x,y)g(y)\,dy\,.$$

The advantage of introducing u_ε, is that the presence of the factor $\eta_\varepsilon(x,y)$ kills the singularity in the integral, as, by (1.7.7), $\eta_\varepsilon(x,y) = 0$ when $||x-y|| \leq \varepsilon$. Moreover, by (1.7.7) again, when $||x-y|| \geq 2\varepsilon$, we have $\eta_\varepsilon(x,y) = 1$, thus, in some sense, u_ε approximates u for small ε. More precisely,

$$|u_\varepsilon(x) - u(x)| = \Big|\int\limits_{\Omega\cap B_{2\varepsilon}(x)} \overline{\phi}(x-y)\big(1-\eta_\varepsilon(x,y)\big)g(y)\,dy\Big|$$

$$\leq \int\limits_{\Omega\cap B_{2\varepsilon}(x)} |\overline{\phi}(x-y)| \sup|g|\,dy \leq \sup|g| \int\limits_{B_{2\varepsilon}(x)} |\overline{\phi}(x-y)|\,dy = \sup|g| \int\limits_{B_{2\varepsilon}(0)} |\overline{\phi}(y)|\,dy \underset{\varepsilon\to 0}{\longrightarrow} 0$$

where we have used (1.7.6) (in the inequality) and Corollary 1.7.2. As the last integral in the previous formula does not depend on x, we have

$$u_\varepsilon \to u \qquad \text{uniformly}\,.$$

By the way, this proves that u is continuous. In fact, every u_ε is continuous, as, in the integral defining it, the factor $\overline{\phi}(x-y)\eta_\varepsilon(x-y)$ is continuous, thus bounded for (x,y) in a bounded set, and g is bounded by hypothesis.

On the other hand, we have

$$\frac{\partial u_\varepsilon}{\partial x_i}(x) = \int\limits_\Omega \frac{\partial}{\partial x_i}\big(\overline{\phi}(x-y)\eta_\varepsilon(x,y)\big)g(y)\,dy$$

as in this case we can differentiate under integral sign, the integrand function being regular. We put

$$v(x) := \int\limits_\Omega \frac{\partial\overline{\phi}}{\partial x_i}(x-y)g(y)\,dy\,.$$

We have

$$\Big|\frac{\partial u_\varepsilon}{\partial x_i}(x) - v(x)\Big| = \Big|\int\limits_{\Omega\cap B_{2\varepsilon}(x)} \frac{\partial}{\partial x_i}\Big(\overline{\phi}(x-y)\big(1-\eta_\varepsilon(x,y)\big)\Big) g(y)\,dy\Big|$$

$$\leq \sup|g| \int\limits_{B_{2\varepsilon}(x)} \Big(\Big|\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\Big| + |\overline{\phi}(x-y)|\frac{2}{\varepsilon}\Big)\,dy \qquad \text{(by (1.7.6) and (1.7.8))}$$

$$\leq \sup|g|\Bigg(\int\limits_{B_{2\varepsilon}(x)} \Big|\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\Big|\,dy + 4\int\limits_{B_{2\varepsilon}(x)} \frac{|\overline{\phi}(x-y)|}{||x-y||}\,dy\Bigg) =$$

$$\sup|g|\Bigg(\int\limits_{B_{2\varepsilon}(0)} \Big|\frac{\partial}{\partial x_i}\overline{\phi}(y)\Big|\,dy + 4\int\limits_{B_{2\varepsilon}(0)} \frac{|\overline{\phi}(y)|}{||y||}\,dy\Bigg) \underset{\varepsilon\to 0}{\longrightarrow} 0$$

by Corollary 1.7.2 and (1.7.1). In conclusion, we have proved that $\dfrac{\partial u_\varepsilon}{\partial x_i} \underset{\varepsilon\to 0}{\longrightarrow} v$ uniformly. As $u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u$ uniformly, by known results on the derivatives of a limit of functions, we have $\dfrac{\partial u}{\partial x_i} = v$, that is the statement of the Theorem. ■

We now want to find the second derivatives of the function u defined in (1.7.4). Actually, the formula defining the second derivatives is not the analog of Theorem 1.7.3, but is more complicated. Such a difference is related to the fact that the first derivatives are summable (Corollary 1.7.2), while the second derivatives are not. Moreover, we have to require an additional hypothesis on g, for example the uniform local Lipshitz property.

Theorem 1.7.4. *Suppose $g : \Omega \to \mathbb{R}$, and*

$$\exists\,\delta > 0,\ \exists\, k > 0 : \ \Big(x, y \in \Omega, ||x-y|| \leq \delta\Big) \Rightarrow |g(x) - g(y)| \leq k||x-y||\,. \tag{1.7.9}$$

Then,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int\limits_{\Omega_0} \frac{\partial^2 \overline{\phi}}{\partial x_i \partial x_j}(x-y)\ \big(g(y)-g(x)\big)\,dy \ - \ g(x)\int\limits_{\partial\Omega_0} \frac{\partial\overline{\phi}}{\partial x_i}(x-y)\nu_j(y)\,dy \tag{1.7.10}$$

for every $x \in \Omega$, where Ω_0 is a connected, bounded, open, div-regular set containing Ω, and g is extended to 0 on $\Omega_0 \setminus \Omega$.

Proof. First, we prove that the first integral in (1.7.10) is defined. We have

$$\Big|\frac{\partial^2 \overline{\phi}}{\partial x_i \partial x_j}(x-y)\ \big(g(y)-g(x)\big)\Big| \leq c_{2,n}||x-y||^{-n} k||x-y||$$

when $y \in B_\delta(x)$ with $\delta > 0$ so small that $\overline{B_\delta(x)} \subseteq \Omega$, by (1.7.3) and (1.7.9). As, moreover, the integrand function is bounded in $\Omega_0 \setminus B_\delta(x)$, by Lemma 1.7.1 the first integral in (1.7.10) is defined. We define η and η_ε as in Theorem 1.7.3. We set

$$v(x) = \frac{\partial u}{\partial x_i}(x)\,,$$

$$v_\varepsilon(x) = \int\limits_{\Omega} \frac{\partial}{\partial x_i}\big(\overline{\phi}(x-y)\big)\eta_\varepsilon(x,y)g(y)\,dy\,.$$

By an argument similar to that used in Theorem 1.7.3, we get that

$$v_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} v \quad \text{uniformly}\,. \tag{1.7.11}$$

On the other hand, we can differentiate in v_ε under integral, as the integrand function has no singularities. Hence

$$\frac{\partial}{\partial x_j}v_\varepsilon(x) = \int\limits_{\Omega_0} \frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\eta_\varepsilon(x,y)\Big)g(y)\,dy$$

$$= \int\limits_{\Omega_0} \frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\eta_\varepsilon(x,y)\Big)\big(g(y)-g(x)\big)\,dy \;+\; g(x)\int\limits_{\Omega_0} \frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\eta_\varepsilon(x,y)\Big)\,dy \tag{1.7.12}$$

where we can integrate on Ω_0 instead of on Ω as g is 0 on $\Omega_0\setminus\Omega$. Note that the integrand in the second integral in the second line of (1.7.12) has the form $\frac{\partial}{\partial x_j}\beta(x-y) = -\frac{\partial}{\partial y_j}\beta(x-y)$, as both $\frac{\partial}{\partial x_i}\overline{\phi}(x-y)$ and, by the definition of η_ε, $\eta_\varepsilon(x,y)$ can be expressed as functions of $x-y$ (see (1.7.1)). Thus such an integral amounts to

$$-\int\limits_{\Omega_0} \frac{\partial}{\partial y_j}\Big(\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\eta_\varepsilon(x,y)\Big)\,dy$$

which, in turn, by the divergence Theorem, amounts to

$$-\int\limits_{\partial\Omega_0} \frac{\partial}{\partial x_i}\overline{\phi}(x-y)\eta_\varepsilon(x,y)\nu_j(y)\,dy = -\int\limits_{\partial\Omega_0} \frac{\partial}{\partial x_i}\overline{\phi}(x-y)\nu_j(y)\,dy$$

for sufficiently small ε, as it suffices to take $\varepsilon < \frac{1}{2}d(x,\partial\Omega_0)$ taking into account that $\eta_\varepsilon(x,y) = 1$ if $||x-y|| \geq 2\varepsilon$. Put now

$$\gamma(x) = \int\limits_{\Omega_0} \frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\Big)\big(g(y)-g(x)\big)\,dy \;-\; g(x)\int\limits_{\partial\Omega_0} \frac{\partial\overline{\phi}}{\partial x_i}(x-y)\nu_j(y)\,dy\,.$$

Then, by proceeding like in Theorem 1.7.3, in view of the previous considerations, we get

$$\Big|\frac{\partial}{\partial x_i} v_\varepsilon(x) - \gamma(x)\Big| =$$

$$\Big|\int\limits_{\Omega_0} \frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\big(1-\eta_\varepsilon(x,y)\big)\Big)\big(g(y)-g(x)\big)\,dy\Big|$$

$$\leq \int\limits_{\Omega_0\cap B_{2\varepsilon}(x)} \Big(\Big|\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\frac{\partial}{\partial x_j}\eta_\varepsilon(x,y)\Big| + \Big|\frac{\partial^2}{\partial x_i\partial x_j}\overline{\phi}(x-y)\Big|\Big)k||x-y||\,dy$$

$$\leq \int\limits_{\Omega_0\cap B_{2\varepsilon}(x)} \Big(\Big|\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\Big|\frac{2}{\varepsilon} + \Big|\frac{\partial^2}{\partial x_i\partial x_j}\overline{\phi}(x-y)\Big|\Big)k||x-y||\,dy$$

by (1.7.6) and (1.7.8), provided

$$\varepsilon < \frac{1}{2}\min\big\{\delta, d(x,\mathbb{R}^n\setminus\Omega)\big\}\,, \tag{1.7.13}$$

so that, if $y\in B_{2\varepsilon}(x)$, then $y\in\Omega$ and, also, we can use (1.7.9). Now, for $y\in B_{2\varepsilon}(x)$, we have $\frac{2}{\varepsilon}||x-y||\leq 4$. Hence,

$$\Big|\frac{\partial}{\partial x_i} v_\varepsilon(x) - \gamma(x)\Big| \leq 4k\Big(\int\limits_{B_{2\varepsilon}(x)} \Big|\frac{\partial}{\partial x_i}\overline{\phi}(x-y)\Big|\,dy + c_{2,n}k\int\limits_{B_{2\varepsilon}(x)} ||x-y||^{1-n}\,dy\Big)$$

$$= 4k\Big(\int\limits_{B_{2\varepsilon}(0)} \Big|\frac{\partial}{\partial y_i}\overline{\phi}(y)\Big|\,dy + c_{2,n}k\int\limits_{B_{2\varepsilon}(0)} ||y||^{1-n}\,dy\Big) \underset{\varepsilon\to 0}{\longrightarrow} 0$$

by (1.7.1) and Corollary (1.7.2). Thus, $\frac{\partial}{\partial x_j} v_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \gamma$ *locally* uniformly. In the present case, the convergence is locally uniform and not (a priori) uniform, as we required that ε satisfies (1.7.13), and, clearly such ε can be chosen independent of x locally, but not globally. In any case, thanks to (1.7.11), this suffices to conclude the proof. ■

Theorem 1.7.5. *If g satisfies (1.7.9), then*

$$\Delta u(x) = g(x) \quad \forall\, x\in\Omega\,.$$

Proof. Let Ω_0 be, e.g., an open ball containing Ω. Then, by Theorem 1.7.4 we have

$$\Delta u(x) = \sum_{i=1}^{n}\frac{\partial^2}{\partial {x_i}^2}u(x) =$$

$$\int\limits_{\Omega_0} \sum_{i=1}^{n} \Big(\frac{\partial^2 \overline{\phi}}{\partial^2 x_i}(x-y)\ \big(g(y)-g(x)\big)\Big)\, dy \ -\ g(x) \int\limits_{\partial\Omega_0} \sum_{i=1}^{n} \Big(\frac{\partial \overline{\phi}}{\partial x_i}(x-y)\nu_i(y)\Big)\, dy$$

$$= \int\limits_{\Omega_0} \Delta\overline{\phi}(x-y)\ \big(g(y)-g(x)\big)\Big)\, dy \ -\ g(x) \int\limits_{\partial\Omega_0} \sum_{i=1}^{n}\ \Big(\frac{\partial \overline{\phi}}{\partial x_i}(x-y)\nu_i(y)\Big)\, dy =$$

$$-\ g(x) \int\limits_{\partial\Omega_0} \sum_{i=1}^{n}\ \Big(\frac{\partial \overline{\phi}}{\partial x_i}(x-y)\nu_i(y)\Big)\, dy$$

as $\overline{\phi}$ is harmonic. Let r be so that $\overline{B}_r(x) \subseteq \Omega$. We then have:

$$-\ g(x) \int\limits_{\partial\Omega_0} \sum_{i=1}^{n}\ \Big(\frac{\partial \overline{\phi}}{\partial x_i}(x-y)\nu_i(y)\Big)\, dy = -\ g(x) \int\limits_{\partial\Omega_0} \mathrm{grad}_x\overline{\phi}(x-y)\cdot\nu(y)\, dy$$

$$= \ g(x) \int\limits_{\partial\Omega_0} \mathrm{grad}_y\overline{\phi}(y-x)\cdot\nu(y)\, dy \qquad \text{by (1.7.1)}$$

$$= g(x) \int\limits_{\partial B_r(x)} \mathrm{grad}_y\overline{\phi}(y-x)\cdot\nu(y)\, dy = g(x)$$

by Corollary 1.3.4 with $A = \Omega_0 \setminus B_r(x)$, and Remark 1.4.2. ∎

We can now prove that, under relatively mild conditions, the problem $P_{f,g}$ has a unique solution.

Corollary 1.7.6. *Suppose at any point of Ω there exists a barrier, for example $\partial\Omega$ of class C^2. Suppose g satisfies (1.7.9). Then, the problem $P_{f,g}$ has a unique solution.*

Proof. We already know that the solution, if exists, is unique. Prove now the existence. Let u be the Newtonian potential defined by (1.7.4). Let $\bar{u} := u|_{\partial\Omega}$. Let v be the solution of $P_{f-\bar{u},0}$, in other words, the solution of the Dirichlet problem on Ω with values $f-\bar{u}$ on $\partial\Omega$. Such a solution does exist by Theorem 1.6.10. Then, clearly, $u+v$ solves $P_{f,g}$. ∎

Remark 1.7.7. Theorem 1.7.4 (as well as, consequently, Theorem 1.7.5 and Corollary 1.7.6) remains valid with essentially the same proof if we replace the Lipshitz property (1.7.9) with the weaker property

$$\exists\,\delta>0,\ \exists\,k>0,\ \exists\,\alpha\in]0,1]:\ \ \Big(x,y\in\Omega, ||x-y||\le\delta\Big) \Rightarrow |g(x)-g(y)|\le k||x-y||^{\alpha}\,.$$

that is, if g is locally uniformly *Hölder continuous.* ∎

If the solution of $P_{f.g}$ exists, we could represent it using the Green function. Namely, in view of Corollary 1.3.5 with $A = \Omega\setminus B_r(y)$ and $v(x) = G(x,y)$,

$$-\int_{\Omega} G(x,y)\Delta u(x)\,dx + \int_{B_r(y)} G(x,y)\Delta u(x)\,dx =$$

$$\int_{\partial\Omega} u(x)\frac{\partial G(x,y)}{\partial x}\cdot\nu(x)\,dx - \int_{\partial B_r(y)} u(x)\frac{\partial G(x,y)}{\partial x}\cdot\nu(x)\,dx + \int_{\partial B_r(y)} G(x,y)\mathrm{grad}u(x)\cdot\nu(x)\,dx$$

where $\frac{\partial G(x,y)}{\partial x}$ stands for $\mathrm{grad}_x G(x,y)$, $y\in\Omega$ and r is so small that $\overline{B_r(y)}\subseteq\Omega$. We have used that $G(x,y)=0$ when $x\in\partial\Omega$, and that G is harmonic with respect to x.

However, such considerations, which use Corollary 1.3.5, are valid if Ω is div-regular and G and u are of class C^2 on an open set containing $\overline{\Omega}$.

We now take the limit for $r\to 0$. We have, in view of (1.5.2) and Corollary 1.3.4,

$$\Big|\int_{\partial B_r(y)} G(x,y)\mathrm{grad}u(x)\cdot\nu(x)\,dx\Big| =$$

$$\Big|\overline{\psi}(r)\int_{\partial B_r(y)} \mathrm{grad}u(x)\cdot\nu(x)\,dx + \int_{\partial B_r(y)} h(x,y)\mathrm{grad}u(x)\cdot\nu(x)\,dx\Big|$$

$$=\Big|\int_{\partial B_r(y)} h(x,y)\mathrm{grad}u(x)\cdot\nu(x)\,dx\Big|$$

$$\leq \max_{x\in\overline{B_{\bar r}(y)}}|h(x,y)|\max_{x\in\overline{B_{\bar r}(y)}}||\mathrm{grad}\,u(x)||\int_{\partial B_r(y)} 1 \underset{r\to 0}{\longrightarrow} 0$$

for $r\leq\bar r$, where $\bar r$ is a positive number such that $\overline{B_{\bar r}(y)}\subseteq\Omega$. Similarly, in the integral

$$\int_{\partial B_r(y)} u(x)\frac{\partial G(x,y)}{\partial x}\cdot\nu(x)\,dx$$

the part with $h(x,y)$ tends to 0 as $r\to 0$, so that, such an integral has the same limit, for $r\to 0$, as

$$\int_{\partial B_r(y)} u(x)\frac{\partial\overline{\phi}(x-y)}{\partial x}\cdot\nu(x)\,dx\,.$$

Finally, the integral $\int_{B_r(y)} G(x,y)\Delta u(x)\,dx$ tends to 0 as $r\to 0$, as the integrand function is summable. In view of Remark 1.4.2, we thus have

$$u(y)=\int_{\Omega} G(x,y)\Delta u(x)\,dx + \int_{\partial\Omega} u(x)\frac{\partial G(x,y)}{\partial x}\cdot\nu(x)\,dx\,.$$

In conclusion, the solution of $P_{f,g}$, if exists is given by

$$u(y) = \int_{\Omega} G(x,y)g(x)\,dx + \int_{\partial\Omega} f(x)\frac{\partial G(x,y)}{\partial x}\cdot \nu(x)\,dx\,,$$

and this formula, as stated before, holds under suitable conditions, related to Corollary 1.3.5.

An important feature of harmonic functions is that they can be characterized in variational terms. Namely, the harmonic functions on Ω are the functions u that minimize the *Dirichlet integral*

$$I_D(u) = \int_D ||\mathrm{grad} u||^2 \tag{1.7.14}$$

among the C^2 functions with the same values on ∂D, for any bounded domain D such that $\overline{D} \subseteq \Omega$.

Theorem 1.7.8. *A function u of class C^2 on Ω is harmonic if and only if for every bounded div-regular domain D such that $\overline{D} \subseteq \Omega$, we have $I_D(u) \le I_D(w)$ for every w of class C^2 on Ω such that $u = w$ on ∂D, where I is the Dirichlet integral defined in (1.7.14).*

Proof. Let D be as in the hypothesis. Put

$$E_f = \left\{u : \Omega \to \mathbb{R} : u \in C^2(\Omega) : u = f \text{ on } \partial D\right\}.$$

for $f : \partial D \to \mathbb{R}$. Note that, for every u and v of class C^2 on Ω we have

$$I_D(u+tv) = t^2 \int_D ||\mathrm{grad} v||^2 \; + 2t \int_D \mathrm{grad} u \cdot \mathrm{grad} v \; + \int_D ||\mathrm{grad} u||^2$$

so that, given a function $\bar{u}$ of class C^2 on Ω, putting $g_v(t) := I_D(\bar{u}+tv)$, with $v \in E_0$, in view of Lemma 1.3.3, we have

$$g_v'(0) = 2\int_D \mathrm{grad}\bar{u} \cdot \mathrm{grad} v = -2\int_D \Delta\bar{u}\; v\,. \tag{1.7.15}$$

Since, as easily verified, g_v is convex, then g_v takes its minimum at 0 if and only if $g_v'(0) = 0$. On the other hand, the set of the functions w of class C^2 on Ω such that $\bar{u} = w$ on ∂D amounts to the set of the functions of the form $\bar{u} + tv$ with $v \in E_0$. Therefore, by (1.7.15), we have $I_D(\bar{u}) \le I_D(w)$ for every w of class C^2 on Ω such that $\bar{u} = w$ on ∂D if and only if $\int_D \Delta\bar{u}\; v = 0$ for every $v \in E_0$, which in turns amounts to $\Delta\bar{u} = 0$ on D, and this concludes the proof. In fact, if there exists $\bar{x} \in D$ such that $\Delta\bar{u}(\bar{x}) \neq 0$, then $\Delta(\bar{u})$ is either positive or negative on a suitable ball B of center $\bar{x}$ contained in D, and we can easily find a function $v \in E_0$, positive on B and null on $D \setminus B$. Consequently, $\int_D \Delta\bar{u}\; v \neq 0$. ■

1.8 Miscellaneous Results on Harmonic Functions

In this Section, we prove some more results, of different types, on harmonic functions. We note that if u is harmonic on Ω, then by Corollary 1.3.5, we have

$$\int_{\Omega} u\,\Delta v = 0 \qquad \forall\, v \in C_c^{\infty}(\Omega)\,. \tag{1.8.1}$$

At first glance, one could suspect that we have to require that Ω is div-regular, but this is not so. In fact, let $\tilde{u}$ be a regular function that amounts to u on $\mathrm{supp}(v)$, and we put $v = 0$ on the complement of Ω. Then, we use Corollary 1.3.5, where A is a ball containing $\mathrm{supp}(v)$, and obtain

$$\int_{\Omega} u\,\Delta v = \int_{supp(v)} u\,\Delta v = \int_{A} \Big(\tilde{u}\,\Delta v - v\Delta\tilde{u}\Big) = 0\,.$$

A continuous function on Ω that satisfies (1.8.1) is said to be a *weak solution* of the Laplace equation on Ω. We are now going to prove that the converse also holds: a weak solution of the Laplace equation is harmonic. Such a result is called *Weyl Lemma*. To prove the Weyl Lemma, we need some preliminary considerations.

Lemma 1.8.1. *If $f \in L^1([a,b])$ and*

$$\int_a^b fg = 0 \tag{1.8.2}$$

for every $g \in C_c^{\infty}(a,b)$, then $f = 0$ a.e. If (1.8.2) holds for every $g \in C_c^{\infty}(a,b)$ such that $\int_a^b g = 0$, then f is constant a.e.

Proof. Let $f \in L^1([a,b])$. Then, the set of $g \in L^{\infty}([a,b])$ satisfying (1.8.2) is a linear subspace of $L^{\infty}([a,b])$. Moreover, if $g_h, g \in L^{\infty}([a,b])$ and $g_h \underset{h\to+\infty}{\longrightarrow} g$ a.e., and $|g_h| \le H$ a.e. for a suitable constant H independent of h (this is the case, in particular, if $g_h \underset{h\to+\infty}{\longrightarrow} g$ uniformly), and moreover, g_h satisfy (1.8.2), then g satisfies (1.8.2) as well. In fact, $fg_h \underset{h\to+\infty}{\longrightarrow} fg$ a.e., and $|fg_h| \le H|f|$ a.e., and we use the dominated convergence Theorem. As every $g \in C_c(a,b)$ is a uniform limit of functions in $C_c^{\infty}(a,b)$, and (1.8.2) by hypothesis holds for every $g \in C_c^{\infty}(a,b)$, (1.8.2) then holds for every $g \in C_c(a,b)$. On the other hand, for every nonempty compact subset K of (a,b), χ_K is the pointwise limit of a sequence of continuous functions with compact support, and with values in $[0,1]$, for example $g_h(x) = \dfrac{d\big(x,[a,b]\setminus K_{\frac{1}{h}}\big)}{d\big(x,[a,b]\setminus K_{\frac{1}{h}}\big)+d(x,K)}$, where

$$K_{\frac{1}{h}} := \Big\{x \in \mathbb{R} : d(x,K) < \frac{1}{h}\Big\}\,.$$

Indeed, if $x \in K$, then $d(x,K) = 0$, hence $g_h(x) = 1$ for every h, and $g_h(x) \underset{h\to+\infty}{\longrightarrow} 1 = \chi_K(x)$. If, on the contrary, $x \notin K$, then, for sufficiently large h, $d(x,K) > \frac{1}{h}$, hence $d\big(x, [a,b] \setminus K_{\frac{1}{h}}\big) = 0$, hence $g_h(x) \underset{h\to+\infty}{\longrightarrow} 0 = \chi_K(x)$. As a consequence, χ_K satisfies (1.8.2) for every nonempty compact subset K of (a,b). Recall that for every measurable set E and for every positive integer h there exists a compact set $K_h \subseteq E$ such that $\mu(E \backslash K_h) < \frac{1}{h}$, and we can and do assume $K_h \subseteq K_{h+1}$. Thus, we have $\chi_{K_h} \underset{h\to+\infty}{\longrightarrow} \chi_E$ on the complement of $E \backslash \Big(\bigcup\limits_{h=1}^{\infty} K_h \Big)$, hence a.e. As a consequence, χ_E satisfies (1.8.2) for every measurable set E, and this in turns implies that the simple functions, linear combinations of characteristic functions, satisfy (1.8.2). Finally, every bounded measurable function, being a uniform limit of simple functions, satisfies (1.8.2). Taking $g = $sign$f$ in (1.8.2), we then have $\int\limits_a^b |f| = 0$, hence $f = 0$ a.e.

Suppose now (1.8.2) holds for every $g \in C_c^\infty(a,b)$ such that $\int\limits_a^b g = 0$. Then, take any $g \in C_c^\infty(a,b)$ and let $g_1 \in C_c^\infty(a,b)$ be such that $\int\limits_a^b g_1 = 1$. Then, since, as easily verified the function $\tilde{g} := g - g_1 \int_a^b g$ is in $C_c^\infty(a,b)$ and $\int\limits_a^b \tilde{g} = 0$, we have $\int\limits_a^b f\tilde{g} = 0$. Hence,

$$\int\limits_a^b g\Big(f - \int\limits_a^b fg_1\Big) = \int\limits_a^b fg - \int\limits_a^b g \int\limits_a^b fg_1 = \int\limits_a^b f\Big(g - g_1 \int\limits_a^b g\Big) = 0\,,$$

and, since this holds for every $g \in C_c^\infty(a,b)$, by the first statement of this Theorem, $f - \int\limits_a^b fg_1 = 0$ a.e., hence $f = c$ a.e., with $c := \int\limits_a^b fg_1$. ■

Theorem 1.8.2 (Weyl Lemma). *If u is continuous on Ω (resp. $u \in L^1(D)$ for every compact subset D of Ω), and (1.8.1) holds, then u is harmonic on Ω (resp. u coincides a.e. with a harmonic function on Ω).*

Proof. Let $\bar{x} \in \Omega$, let $\bar{R} > 0$ be such that $\overline{B_{\bar{R}}(\bar{x})} \subseteq \Omega$, and let $R \in]0, \bar{R}[$. Let h be a function of class C^∞ on $\mathbb{R}$ such that

$$h = \begin{cases} c & \text{on } [0,\varepsilon] \\ 0 & \text{on } [R, +\infty[\end{cases}$$

where c is a suitable constant and $0 < \varepsilon < R$. Let $v(x) = h(||x - \bar{x}||)$. Clearly: $v = 0$ on $\mathbb{R}^n \setminus B_R(\bar{x})$, $v = h(\varepsilon)$ on $B_\varepsilon(\bar{x})$. We easily get (cf. Section 1.3)

$$\Delta v(x) = \frac{h''\big(||x-\bar{x}||\big)||x-\bar{x}||^3 + (n-1)h'\big(||x-\bar{x}||\big)||x-\bar{x}||^2}{||x-\bar{x}||^3}\,.$$

Hence

$$\int_{\Omega} u\,\Delta v = \int_{B_R(\bar{x})\setminus B_\varepsilon(\bar{x})} u(x)\frac{h''\big(||x-\bar{x}||\big)||x-\bar{x}||^3 + (n-1)h'\big(||x-\bar{x}||\big)||x-\bar{x}||^2}{||x-\bar{x}||^3}\,dx =$$

$$\int_{\varepsilon}^{R}\Bigg(\int_{\partial B_t(\bar{x})} u(x)\frac{h''\big(t\big)t^3 + (n-1)h'\big(t\big)t^2}{t^3}\,dx\Bigg)\,dt =$$

$$\int_{\varepsilon}^{R}\frac{d}{dt}\big(h'(t)t^{n-1}\big)\;\Big(\frac{1}{t^{n-1}}\int_{\partial B_t(\bar{x})} u(x)\,dx\Big)\;dt$$

hence, by hypothesis

$$\int_{\varepsilon}^{R}\frac{d}{dt}\big(h'(t)t^{n-1}\big)\;\Big(\frac{1}{t^{n-1}}\int_{\partial B_t(\bar{x})} u(x)\,dx\Big)\;dt = 0\,. \tag{1.8.3}$$

We have $\int_{\varepsilon}^{R}\frac{d}{dt}\big(h'(t)t^{n-1}\big)\;dt = h'(R)R^{n-1} - h'(\varepsilon)\varepsilon^{n-1} = 0$. However, what we really need is the converse, that is, given $\eta \in]0,\bar{R}[$ if $\alpha \in C_c^\infty(\eta,\bar{R})$, and supp $\alpha \subseteq [\varepsilon, R]$, $\eta < \varepsilon < R$, and $\int_{\varepsilon}^{R}\alpha = 0$, then there exists h as above such that

$$\alpha(t) = \frac{d}{dt}\Big(h'(t)t^{n-1}\Big)\,. \tag{1.8.4}$$

In fact, let $A(t) = \int_{\varepsilon}^{t}\alpha(s)\,ds$. Clearly, $A(\varepsilon) = A(R) = 0$, and A is 0 on the complement of $[\varepsilon, R]$. Then, if (1.8.4) holds, we have $A(t) = h'(t)t^{n-1}$, and we finally obtain the function h satisfying (1.8.4), namely,

$$h(t) = \int_{\varepsilon}^{t}\frac{A(s)}{s^{n-1}}\,ds - \int_{\varepsilon}^{R}\frac{A(s)}{s^{n-1}}\,ds\,.$$

By (1.8.3), we can use Lemma 1.8.1 with $[a,b] = [\eta,\bar{R}]$ and conclude that

$$\frac{1}{t^{n-1}}\int_{\partial B_t(\bar{x})} u(x)\,dx = c$$

with c constant, a.e. $t \in [\eta,\bar{R}]$ for every $\eta \in]0,\bar{R}[$, hence for almost every $t \in [0,\bar{R}]$, or also

$$\frac{1}{\int_{\partial B_t(\bar{x})} 1}\int_{\partial B_t(\bar{x})} u(x)\,dx = c'$$

with $c' = cd_n$ a.e. $t \in [0, \bar{R}]$. The device of using α in $C_c^\infty(\eta, \bar{R})$ instead of in $C_c^\infty(0, \bar{R})$ is necessary in the L^1 version in order to apply Lemma 1.8.1, as in general, the map $t \mapsto \frac{1}{t^{n-1}} \int\limits_{\partial B_t(\bar{x})} u(x)\, dx$ is not L^1 on $(0, \bar{R})$, but it is L^1 on $(\eta, \bar{R})$. In fact, it suffices to apply Corollary 1.2.2 (or more precisely, a version of it, with u in L^1 instead of continuous), to get that the integral $\int\limits_\eta^{\bar{R}} \left(\frac{1}{t^{n-1}} \int\limits_{\partial B_t(\bar{x})} |u(x)|\, dx \right) dt$ is finite. Then, we integrate with respect to $t \in [0, \bar{R}]$ and by the same argument as in proof of Lemma 1.4.3 we get

$$c' = \frac{1}{\mu(B_r(\bar{x}))} \int\limits_{B_r(\bar{x})} u\,, \tag{1.8.5}$$

for every $r \in]0, \bar{R}[$. Now, if we take the limit of the left-hand side of (1.8.5), we get $c' = u(\bar{x})$, hence

$$u(\bar{x}) = \frac{1}{\mu(B_r(\bar{x}))} \int\limits_{B_r(\bar{x})} u$$

provided u is continuous. In such a case, by Theorem 1.4.8, u is harmonic. If, instead, u is only assumed to be in $L^1(D)$ for every compact subset D of Ω, then, by a known result in measure theory, the left-hand side tends to $u(\bar{x})$ for almost every $\bar{x} \in \Omega$, as $r \to 0$. Hence, setting

$$\tilde{u}(x) = \frac{1}{\mu(B_r(x))} \int\limits_{B_r(\bar{x})} u \tag{1.8.6}$$

where $r > 0$ is such that $\overline{B_r(x)} \subseteq \Omega$, by (1.8.5) the definition of $\tilde{u}$ is independent of r and $u = \tilde{u}$ a.e.. Moreover, in the integral in (1.8.6) we can replace u with $\tilde{u}$, and, using Theorem 1.4.8 again, $\tilde{u}$ is harmonic. ■

We are now going to prove that the harmonic functions are analytic. Firstly, we recall the definition of analytic function. A multiindex j is an n-tuple $(j_1, ..., j_n)$ of nonnegative integers, and we put $|j| = \sum\limits_{h=1}^n j_h$. By definition we put $a^j = \prod\limits_{h=1}^n a_h^{j_h}$ for every $a \in \mathbb{R}^n$. Here, by convention, $0^0 = 1$. We say that a function u defined on Ω is analytic if for every $\bar{x} \in \Omega$ there exist coefficients c_j corresponding to every multiindex j and a neighbourhood U of $\bar{x}$ such that

$$u(x) = \sum_{n=0}^{+\infty} \sum_{|j|=n} c_j (x - \bar{x})^j$$

for every $x \in U$.

Theorem 1.8.3. *If u is harmonic on Ω, then u is analytic on Ω.*

Proof. We prove that, for every $y \in \Omega$, u is analytic around $y \in \Omega$. We can and do assume $y = 0$, as in the general case we can replace $u(x)$ with $v(x) = u(x+y)$ and prove that this new function, which is clearly harmonic, is analytic around 0, so that u is analytic around y. In order to prove that u is analytic around 0, recalling formula (1.5.6), the first step consists in proving that the function w, defined by

$$w(y) = \int_{\partial B_R(0)} u(x) \, \frac{1}{||x-y||^n} \, dx \,,$$

where r is such that $\overline{B_R(0)} \subseteq \Omega$ is analytic around 0. Note that

$$\frac{1}{||x-y||^n} = \Big(||x-y||^2\Big)^{-\dfrac{n}{2}} = \Big(||x||^2 + ||y||^2 - 2x \cdot y\Big)^{-\dfrac{n}{2}} = ||x||^{-n}\Big(1 + \frac{||y||^2}{||x||^2} - 2\frac{x \cdot y}{||x||^2}\Big)^{-\dfrac{n}{2}}$$

so that, using the Taylor expansion $(1+z)^\alpha = \sum\limits_{k=0}^{+\infty} c_k z^k$, valid when $|z| < 1$, we get

$$\frac{1}{||x-y||^n} = ||x||^{-n} \sum_{k=0}^{+\infty} c_k \Big(\frac{||y||^2}{||x||^2} - 2\frac{x \cdot y}{||x||^2}\Big)^k = ||x||^{-n} \sum_{k=0}^{+\infty} c_k \Big(\frac{||y||^2 - 2x \cdot y}{R^2}\Big)^k \tag{1.8.7}$$

when $x \in \partial B_R(0)$, and $\Big| ||y||^2 - 2x \cdot y \Big| < R^2$. Hence,

$$|u(x)| \frac{1}{||x-y||^n} \le R^{-n} \max_{\partial B_R(0)} |u| \sum_{k=0}^{+\infty} c_k \Big(\frac{r^2 + 2Rr}{R^2}\Big)^k := K$$

when $||y|| < r$, where $r > 0$ is such that

$$r^2 + 2rR < R^2 \,. \tag{1.8.8}$$

Taking into account that $||y||^2 = \sum\limits_{i=1}^{n} y_i^2$ and $x \cdot y = \sum\limits_{i=1}^{n} x_i y_i$, by expanding the powers, and then rearranging the terms, in (1.8.7), we have

$$|u(x)| \frac{1}{||x-y||^n} = \sum_{n=0}^{+\infty} \sum_{|j|=n} \alpha_j(x) y^j \tag{1.8.9}$$

where α_j are continuous functions and the series of the absolute values of the summands of the right-hand side in (1.8.9) is not greater than K. By a simple application of the dominated convergence Theorem, we can then integrate the right-hand side in (1.8.9), exchanging the sum and the integral, thus

$$w(y) = \int_{\partial B_R(0)} u(x) \, \frac{1}{||x-y||^n} \, dx = \sum_{n=0}^{+\infty} \sum_{|j|=n} \Big(\int_{\partial B_R(0)} \alpha_j(x) \, dx \Big) \, y^j \tag{1.8.10}$$

when $|y| < r$ and r satisfies (1.8.8), hence w is analytic around 0 as claimed. By (1.5.6) we have $u(y) = \frac{R^2 - ||y||^2}{d_n R} w(y)$, and using the expression of w in (1.8.10), after performing the multiplication and rearranging the terms we can write u as an analytic function. ∎

The last result we are going to give about harmonic functions concerns the possibility of extending a harmonic function defined of the upper half-space on all of $\mathbb{R}^n$. Such a result is called *reflection principle.*

Theorem 1.8.4. *Let u be a continuous function defined on the half-space $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n \geq 0\}$, such that u is harmonic on the interior of $\mathbb{R}^n_+$, and $u = 0$ on $\partial\mathbb{R}^n_+ \big(= \{x \in \mathbb{R}^n : x_n = 0\}\big)$. Then, the function $\bar{u}$ defined on $\mathbb{R}^n$ by*

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^n_+ \\ -u(\pi(x)) & \text{if } x \notin \mathbb{R}^n_+ \end{cases},$$

where π is the symmetry with respect to $\partial\mathbb{R}^n_+$, in other words, $\pi(x)_i = \begin{cases} x_i & \text{if } i < n \\ -x_i & \text{if } i = n \end{cases}$, is harmonic on $\mathbb{R}^n$.

Proof. Clearly, $\bar{u}$ is continuous, hence by Theorem 1.4.8 and Remark 1.5.2 it suffices to prove that for any $\bar{x} \in \mathbb{R}^n$, there exists $\bar{r} > 0$ such that, if $0 < r < \bar{r}$, then $\bar{u}$ satisfies the mean value property (1.4.5). If $\bar{x}_n > 0$, that is $\bar{x}$ is an interior point of $\mathbb{R}^n_+$, as $\bar{u}$ coincides with the harmonic function u on $\mathbb{R}^n_+$, it suffices to take $\bar{r}$ such that $\overline{B_{\bar{r}}(\bar{x})} \subseteq \mathbb{R}^n_+$. If $\bar{x}_n < 0$, by a symmetry argument and the definition of $\bar{u}$, we have $\bar{u}(\bar{x}) = -\bar{u}(\pi(\bar{x}))$, and, taking $r < d(x, \mathbb{R}^n_+)$,

$$\frac{1}{\mu(B_r(\bar{x}))} \int_{B_r(\bar{x})} u = -\frac{1}{\mu\big(B_r(\pi(\bar{x}))\big)} \int_{B_r(\pi(\bar{x}))} u$$

so that the mean value property for $\bar{x}$ follows from the analogous property for $\pi(\bar{x})$ which lies in the interior of $\mathbb{R}^n_+$. Finally, if $\bar{x}_n = 0$, then $\bar{u}(\bar{x}) = 0$ by hypothesis, and

$$\frac{1}{\mu(B_r(\bar{x}))} \int_{B_r(\bar{x})} u = 0$$

by the definition of $\bar{u}$ and a symmetry argument. ∎

1.9. Other Kinds of Partial Differential Equations

We now consider a generalization of the Laplace operator, namely,

$$L(u) = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x) \tag{1.9.1}$$

where we suppose for simplicity that the functions $a_{i,j}, b_i, c$ are continuous, and $a_{i,j} = a_{j,i}$, that is the matrix $A(x)$ with coefficients $a_{i,j}(x)$ is symmetric. The Laplacian corresponds to the case where the matrix $A(x)$ is the identity, and $b_i = c = 0$. The matrix $A(x)$, being symmetric, has a basis of eigenvectors (of course, in general depending on x), and has all eigenvalues in $\mathbb{R}$. We denote them by $\lambda_1(x)$, $\lambda_2(x)$, ..., $\lambda_n(x)$, ordered increasingly, i.e., $\lambda_1(x) \le \lambda_2(x) \le ... \le \lambda_n(x)$. The operator L is said to be elliptic if $\lambda_1(x) > 0$ for all $x \in \Omega$, that is, if $A(x)$ is positive definite for every $x \in \Omega$. Note that by known results in matrix theory, we have

$$\lambda_1(x)||\xi||^2 \le \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \le \lambda_n(x)||\xi||^2 . \tag{1.9.2}$$

An elliptic partial differential equations is an equation of the form $L(U) = 0$ where L is elliptic. The elliptic P.D.E. have properties similar to those of the Laplace equation. Here, we will only discuss about the maximum principle and its consequences. First, I recall a standard result in matrix theory.

Lemma 1.9.1. *If A and B are symmetric positive semidefinite matrices of order n, then the trace of the product AB is nonnegative.*

Proof. As the trace is invariant with respect to a change of basis, we can suppose that the matrix B is diagonal with (i,i) entries equal to $b_{i,i} \ge 0$, as they are the eigenvalues of B. Let $a_{i,j}$ be the (i,j) entries of the matrix B. As $a_{i,i} = A(e_i) \cdot e_i$, we have $a_{i,i} \ge 0$. Then, the trace of AB amounts to $\sum_{i=1}^{n} a_{i,i}b_{i,i} \ge 0$. ■

Theorem 1.9.2. *Let Ω be bounded. Let L be an elliptic operator as in (1.9.1) with $c = 0$, and suppose there exists $c \ge 0$ such that*

$$|b_i(x)| \le c\lambda_1(x) \quad \forall\, x \in \Omega . \tag{1.9.3}$$

Then, if u is a continuous function on $\overline{\Omega}$ such that $L(u) \ge 0$ on Ω, we have $\max\limits_{\overline{\Omega}} u = \max\limits_{\partial\Omega} u$.

Proof. *First Step.* Suppose for the moment $L(u) > 0$ on Ω and prove that u does not attain a maximum value on Ω. Suppose by contradiction $\bar{x} \in \Omega$ and $u(\bar{x}) = \max u$. Then, $\text{grad}(u)(\bar{x}) = 0$, and

$$-L(u)(\bar{x}) = -\sum_{i,j=1}^{n} a_{i,j}(\bar{x})\frac{\partial^2 u(\bar{x})}{\partial x_i \partial x_j} = tr(C)$$

where C is the product of the matrix $A(x)$, which is symmetric, positive definite by hypothesis and the opposite of the Hessian matrix of u at $\bar{x}$, which is symmetric, and positive semidefinite as it is the opposite of the Hessian matrix at a maximum point. By Lemma 1.9.1, we have $L(u)(\bar{x}) \le 0$, a contradiction. It follows that in such a case the maximum of u is attained at a boundary point, and the Theorem is proved.

In the general case, put $v(x) = e^{dx_1}$, so that we have $L(v)(x) = \big(a_{1,1}(x)d^2 + b_1(x)d\big)v(x)$. As $a_{1,1}(x) = \big(A(x)e_1\big) \cdot e_1 \ge \lambda_1(x)$ by (1.9.2), in view of (1.9.3) we then have

$$L(v)(x) \geq \big(\lambda_1(x)d^2 - cd\lambda_1(x)\big)v(x) = d\lambda_1(x)v(x)(d-c) > 0 \tag{1.9.4}$$

provided $d > c$. For $\varepsilon > 0$, we now put $u_\varepsilon = u + \varepsilon v$, so that by the hypothesis and (1.9.4), $L(u_\varepsilon) = L(u) + \varepsilon L(v) > 0$ on Ω. As, clearly $u \leq u_\varepsilon \leq u + \varepsilon \max\limits_{\overline{\Omega}} v$ on $\overline{\Omega}$, by the first step we have

$$\max_{\overline{\Omega}} u = \lim_{\varepsilon\to 0} \max_{\overline{\Omega}} u_\varepsilon = \lim_{\varepsilon\to 0} \max_{\partial\Omega} u_\varepsilon = \max_{\partial\Omega} u$$

and the Theorem is proved in the general case. ∎

Corollary 1.9.3. *In the same hypothesis on L as in Theorem 1.9.2, if u is a continuous function on $\overline{\Omega}$ such that $L(u) = 0$ on Ω, we have $\max\limits_{\overline{\Omega}} u = \max\limits_{\partial\Omega} u$ and $\min\limits_{\overline{\Omega}} u = \min\limits_{\partial\Omega} u$.*

Proof. It suffices to apply Theorem 1.9.2 to the functions u and $-u$. ∎

Corollary 1.9.4. *In the same hypothesis on L as in Theorem 1.9.2, the problem*

$$\begin{cases} L(u) = g & \text{on } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

has at most one solution for every g continuous on Ω and f continuous on $\partial\Omega$.

Proof. If we are given two solutions u_1 and u_2, let $u = u_1 - u_2$. Then u satisfies

$$\begin{cases} L(u) = 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

so that, by Corollary 1.9.2, $u = 0$ on $\overline{\Omega}$ and $u_1 = u_2$. ∎

We will now give an outline of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad c > 0\,. \tag{E}$$

It can be easily verified that every function u of the form

$$u(x,t) = f(x+ct) + g(x-ct) \tag{1.9.5}$$

with f and g of class C^2 on $\mathbb{R}$ is a solution of (E) on $\mathbb{R}^2$. We will now prove that every C^2 solution of (E) has the form of (1.9.5) with f and g of class C^2. In fact, by a change of coordinates, we can write every u of class C^2 on $\mathbb{R}^2$ as $u(x,t) = w(x+ct, x-ct)$ with w of class C^2. Put $w = w(\xi,\eta)$. A simple verification shows

$$\frac{\partial u}{\partial t}(x,t) = c\frac{\partial w}{\partial \xi}(x+ct, x-ct) - c\frac{\partial w}{\partial \eta}(x+ct, x-ct)\,,$$

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 w}{\partial \xi^2}(x+ct, x-ct) + c^2 \frac{\partial^2 w}{\partial \eta^2}(x+ct, x-ct) - 2c^2 \frac{\partial^2 w}{\partial \xi \partial \eta}(x+ct, x-ct)\,,$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 w}{\partial \xi^2}(x+ct, x-ct) + \frac{\partial^2 w}{\partial \eta^2}(x+ct, x-ct) + 2 \frac{\partial^2 w}{\partial \xi \partial \eta}(x+ct, x-ct)\,,$$

and (E) thus amounts to $\dfrac{\partial^2 w}{\partial \xi \partial \eta}(x+ct, x-ct) = 0$ for all $(x,t) \in \mathbb{R}^2$, that is

$$\frac{\partial^2 w}{\partial \xi \partial \eta}(\xi, \eta) = 0 \quad \forall\, (\xi, \eta) \in \mathbb{R}^2\,. \tag{1.9.6}$$

By integrating in (1.9.6), first with respect to η, then with respect to ξ, we get $\dfrac{\partial w}{\partial \xi}(\xi, \eta) = a(\xi)$ and $w(\xi, \eta) = f(\xi) + g(\eta)$, where f is a primitive of a, thus $u(x,t) = w(x+ct, x-ct) = f(x+ct) + g(x-ct)$, as claimed. We now want to solve the Cauchy problem relative to the wave equation.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) \\ u(x,0) = \alpha(x) \\ \dfrac{\partial u}{\partial t}(x,0) = \beta(x) \end{cases} \tag{P}$$

where α and β are prescribed functions defined on $\mathbb{R}$, α of class C^2 and β of class C^1. We have $u(x,t) = f(x+ct) + g(x-ct)$, so that

$$\alpha(x) = f(x) + g(x), \quad \beta(x) = c\big(f'(x) - g'(x)\big)\,,$$

$$B(x) = c\big(f(x) - g(x)\big) + k$$

where B is a primitive of β, and k is a constant. We get

$$\begin{cases} f(x) + g(x) = \alpha(x) \\ f(x) - g(x) = \frac{B(x)-k}{c} \end{cases}$$

$$\begin{cases} f(x) = \dfrac{1}{2}\Big(\alpha(x) + \dfrac{B(x)-k}{c}\Big) \\ g(x) = \dfrac{1}{2}\Big(\alpha(x) - \dfrac{B(x)-k}{c}\Big) \end{cases}$$

so that the solution of (P) is given by

$$u(x,t) = f(x+ct) + g(x-ct) = \frac{1}{2}\Big(\alpha(x+ct) + \alpha(x-ct)\Big) + \frac{1}{2c} \int_{x-ct}^{x+ct} \beta(s)\, ds\,.$$

Note that the Cauchy problem is an analog of the Cauchy problem for ordinary equations.