1. The Isoperimetric Problem

We want to prove that the circle is the figure with maximum area among those with given perimeter. More precisely, we are going to prove that, given a piecewise regular[1] C^1 Jordan curve γ in $\mathbb{R}^2$ of length L, then the area A of the bounded region enclosed by γ is not greater than the area of the circle of perimeter L. The result is valid in the more general case where the hypothesis *piecewise regular* C^1 is replaced by *continuous*, but the proof is more complicated. First of all, we evaluate the area S of the circle of perimeter L. The radius is given by $\frac{L}{2\pi}$, thus $S = \pi(\frac{L}{2\pi})^2 = \frac{L^2}{4\pi}$. Hence, we have to prove

$$L^2 \geq 4\pi A \tag{1.1}$$

In order to prove (1.1), we recall some properties of Fourier series.

Theorem 1.1. *If f is a Riemann integrable function on $[-\pi, \pi]$, then*

$$\int_{-\pi}^{\pi} f^2(t)\, dt = \pi\Big(\frac{a_0^2}{2} + \sum_{n=1}^{+\infty}(a_n^2 + b_n^2)\Big)$$

where a_n, $n = 0, 1, 2...$ and b_n, $n = 1, 2, 3, ...$ are the Fourier coefficients of f defined by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\, dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\, dt$$

Theorem 1.2. *If f and g are Riemann integrable functions on $[-\pi, \pi]$, then*

$$\int_{-\pi}^{\pi} f(t)g(t)\, dt = \pi\Big(\frac{a_0 c_0}{2} + \sum_{n=1}^{+\infty}(a_n c_n + b_n d_n)\Big)$$

where a_n, $n = 0, 1, 2...$ and b_n, $n = 1, 2, 3, ...$ are the Fourier coefficients of f, and c_n, $n = 0, 1, 2...$ and d_n, $n = 1, 2, 3, ...$ are the Fourier coefficients of g.

Theorem 1.3. *If f is a piecewise C^1 function on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$, and a_n, $n = 0, 1, 2...$ and b_n, $n = 1, 2, 3, ...$ are the Fourier coefficients of f, and c_n, $n = 0, 1, 2...$ and d_n, $n = 1, 2, 3, ...$ are the Fourier coefficients of f', we have $c_n = nb_n$, $d_n = -na_n$.*

Theorem 1.1 is a known result in Fourier series. Who is familiar only with the case where f is piecewise C^1, can restrict our considerations to the case where the curve γ is piecewise C^2. Note that under the hypothesis of Theorem 1.1 we are not sure that the Fourier series of f converges to f (pointwise), but nevertheless the Fourier coefficients can be however

[1] regular means $\gamma' \neq 0$

defined. Theorem 1.2 follows from Theorem 1.1, as in the hypothesis of Theorem 1.2 $f+g$ is Riemann integrable so that

$$2\int_{-\pi}^{\pi} f(t)g(t)\,dt = \int_{-\pi}^{\pi} (f(t)+g(t))^2 - f(t)^2 - g(t)^2\,dt =$$

$$\int_{-\pi}^{\pi} (f(t)+g(t))^2\,dt - \int_{-\pi}^{\pi} f(t)^2\,dt - \int_{-\pi}^{\pi} g(t)^2\,dt =$$

$$\pi\Big(\frac{(a_0+c_0)^2}{2} + \sum_{n=1}^{+\infty}((a_n+c_n)^2+(b_n+d_n)^2)\Big)$$

$$-\pi\Big(\frac{a_0^2}{2} + \sum_{n=1}^{+\infty}(a_n^2+b_n^2)\Big) - \pi\Big(\frac{c_0^2}{2} + \sum_{n=1}^{+\infty}(c_n^2+d_n^2)\Big)$$

$$= \pi\Big(a_0c_0 + \sum_{n=1}^{+\infty}(2a_nc_n + 2b_nd_n)\Big)\,.$$

Theorem 1.3 directly follows from the definition of the Fourier coefficients using a partial integration, taking into account that the by the hypothesis, the limit terms at $-\pi$ and at π are equal. We are now going to prove (1.1). Let $\gamma = (x,y)$. We can assume that γ is parametrized by arc length, in particular $\gamma : [0,L] \to \mathbb{R}^2$. In order to use the theory of Fourier series, we reparametrize it in such a way that it is defined on $[-\pi,\pi]$. Namely, let $\alpha : [-\pi,\pi] \to [0,L]$ be defined as $\alpha(t) = \frac{L}{2\pi}(t+\pi)$, and let $\tilde{\gamma} = \gamma \circ \alpha$. Then $\tilde{\gamma}$ has the same image and the same length as γ. Moreover, putting $\tilde{\gamma} = (\tilde{x},\tilde{y})$, we have

$$||\tilde{\gamma}'(t)|| = ||\big(x'(\alpha(t))\alpha'(t), y'(\alpha(t))\alpha'(t)\big)|| = ||\frac{L}{2\pi}\big(x'(\alpha(t)), y'(\alpha(t))\big)||$$

$$= \frac{L}{2\pi}||\big(x'(\alpha(t)), y'(\alpha(t))\big)|| = \frac{L}{2\pi}\,,$$

the last equality depending on the fact that γ is parametrized by arc length. We have

$$\int_{-\pi}^{\pi} \big(\tilde{x}'(t)^2 + \tilde{y}'(t)^2\big)\,dt = \int_{-\pi}^{\pi} ||\tilde{\gamma}'(t)^2||\,dt = \int_{-\pi}^{\pi} \frac{L^2}{4\pi^2}\,dt = \frac{L^2}{2\pi}\,,$$

hence

$$L^2 = 2\pi\int_{-\pi}^{\pi} \big(\tilde{x}'(t)^2 + \tilde{y}'(t)^2\big)\,dt\,.$$

Let now a_n, b_n be the Fourier coefficients of $\tilde{x}$, c_n, d_n be the Fourier coefficients of $\tilde{y}$. As γ is a Jordan curve we have $\gamma(0) = \gamma(L)$ and by definition of $\tilde{\gamma}$, $\tilde{\gamma}(-\pi) = \tilde{\gamma}(\pi)$, so that $\tilde{x}(-\pi) = \tilde{x}(\pi)$, $\tilde{y}(-\pi) = \tilde{y}(\pi)$. Hence, we can apply Theorem 1.3 and deduce that the Fourier coefficients of $\tilde{x}'$ are nb_n and $-na_n$. Using Theorem 1.1 we thus have

$$\int_{-\pi}^{\pi} \tilde{x}'(t)^2\,dt = \pi\sum_{n=1}^{+\infty} n^2(a_n^2 + b_n^2)$$

and by similar considerations

$$\int_{-\pi}^{\pi} \tilde{y}'(t)^2\,dt = \pi \sum_{n=1}^{+\infty} n^2(c_n^2 + d_n^2),$$

and, in conclusion

$$L^2 = 2\pi^2 \sum_{n=1}^{+\infty} n^2(a_n^2 + b_n^2 + c_n^2 + d_n^2)\,. \tag{1.2}$$

Next, we evaluate A using the formula $A = \int_{-\pi}^{\pi} \tilde{x}(t)\tilde{y}'(t)\,dt$, consequence of the Green formula. Using Theorems 1.2 and 1.3 we deduce

$$A = \pi \sum_{n=1}^{+\infty} n(a_n d_n - b_n c_n)\,.$$

Using (1.2) we get

$$\begin{aligned} L^2 - 4\pi A &= 2\pi^2 \Big(\sum_{n=1}^{+\infty} n^2(a_n^2 + b_n^2 + c_n^2 + d_n^2) - 2n a_n d_n + 2n b_n c_n \Big) \\ &= 2\pi^2 \Big(\sum_{n=1}^{+\infty} (n a_n - d_n)^2 + (n b_n + c_n)^2 + (n^2 - 1)(c_n^2 + d_n^2) \Big) \geq 0 \end{aligned}$$

and (1.1) is proved. We also note that the equality holds in (1.1) if and only if the equality holds in the previous inequality, if and only if we have $c_n = d_n = 0$ for $n \geq 2$, $d_n = n a_n$ and $c_n = -n b_n$ for every $n \geq 1$, and this occurs when $a_n = b_n = c_n = d_n = 0$ for every $n \geq 2$, $a_1 = d_1$, $b_1 = -c_1$. Since $\tilde{x}$ and $\tilde{y}$ are piecewise C_1, thus in particular continuous, they amount to the the sum of their Fourier series, hence

$$\tilde{x}(t) = \frac{a_0}{2} + a_1 \cos t + b_1 \sin t \qquad \tilde{y}(t) = \frac{c_0}{2} - b_1 \cos t + a_1 \sin t$$

which represents the equation of a circle[2]. It follows that not only the circle is the plane Jordan curve of given length enclosing the maximum area but also that it is the unique curve having such a property. The solution presented here is due to Hurwitz.

2. The Ascoli-Arzelà Theorem

We start by recalling some base facts about compactness. In order to simplify the presentation, we restrict our considerations to Hausdorff topological spaces. We recall that a (Hausdorff) topological space X is compact if for every family of open subsets U_i of X,

[2] Indeed, setting $r = \sqrt{a_1^2 + b_1^2}$, we have $\left(\frac{a_1}{r}\right)^2 + \left(\frac{b_1}{r}\right)^2 = 1$, hence there exists $\overline{t} \in \mathbb{R}$ so that $\frac{a_1}{r} = \cos \overline{t}$, $\frac{b_1}{r} = \sin \overline{t}$. We thus easily see that $\tilde{x}(t) = \frac{a_0}{2} + r\cos(t - \overline{t})$, $\tilde{y}(t) = \frac{c_0}{2} + r\sin(t - \overline{t})$.

$i \in I$ such that $X = \bigcup_{i \in I} U_i$ there exist $i_1, ..., i_m \in I$ such that $X = U_{i_1} \cup U_{i_2} \cup ... \cup U_{i_m}$. We recall that if X is a metric space then X is compact if and only if every sequence in X has a convergent (in X) subsequence. As a consequence, every compact metric space is complete (recall that in any metric space a *Cauchy* sequence having a convergent subsequence is convergent). We recall that if X is a subset of $\mathbb{R}^n$ then X is compact if and only if it is both closed and bounded. It follows that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. Instead, when X is subset of an infinite dimensional normed space, if X is compact, then it is both closed and bounded, but the converse does not hold, i.e., a closed-and-bounded set is not necessarily compact.

The Ascoli-Arzelà Theorem is related to the problem of what subsets of the space of the continuous functions from a topological space A to $\mathbb{R}$ with the norm $||f|| = \sup |f(x)|$, are compact. More precisely, under what conditions we can state that a sequence of continuous functions from X to $\mathbb{R}$ has a uniformly convergent subsequence. We can more generally suppose that the functions are valued in a metric space (Y, d). We need some preliminary definitions. We recall that if f_n is a sequence of functions from a topological space X with values in a complete metric space, then f_n is uniformly convergent if and only if it is uniformly Cauchy, that is, for every $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that, if $n \geq \nu$, $m \geq \nu$ then $d(f_n(x), f_m(x)) < \varepsilon$ for every $x \in X$.

Definition 2.1. *We say that a sequence f_n of functions from a Hausdorff topological space X into a metric space Y are equicontinuous if for every $x \in X$ and for every $\varepsilon > 0$ there exists a neighborhood U of x such that for every $y \in U$ and for every n we have $d(f_n(x), f_n(y)) < \varepsilon$.*

Note that if f_n are equicontinuous then every f_n is continuous, but the converse is not true, since in the previous definition we require that the neighborhood U does not depend on n. Note also that, since every neighborhood of x contains an open set containing x, by the definition of a neighborhood, in Def. 2.1 we can suppose that U is open. If X is a metric space with a metric d' then the previous definitions can be expressed in terms of ε and δ, i.e., for every $x \in X$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in X$ such that $d'(x, y) < \delta$ and for every n we have $d(f_n(x), f_n(y)) < \varepsilon$. If X is a metric space with a metric d', we say that f_n are uniformly equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ such that $d'(x, y) < \delta$ and for every n we have $d(f_n(x), f_n(y)) < \varepsilon$. In other words, δ is also independent of x. Clearly, if f_n are uniformly equicontinuous then f_n are equicontinuous. It is possible to prove that also the converse holds if X is compact. The argument of the proof is analogous to that used for proving that a continuous function on a compact set is uniformly continuous.

Theorem 2.2 (Ascoli-Arzelà Theorem). *If f_n are equicontinuous functions from a compact topological space X to a compact metric space Y, then there exists a subsequence of f_n uniformly convergent on X.*

Proof. For each $\varepsilon > 0$ and for each $x \in X$ let $U_{\varepsilon,x}$ be an open set in X containing x

such that for each $y \in U_{\varepsilon,x}$ and for every n, we have $d(f_n(x), f_n(y)) < \varepsilon$. Such a set $U_{\varepsilon,x}$ exists by the assumption that f_n are equicontinuous. As, for a given $s = 1, 2, 3, ...$ the sets $U_{1/s,x}$, $x \in X$ are open sets whose union is X, X being compact there exists a finite subset $A_s = \{x(i, s) : i = 1, ...m(s)\}$ of X such that

$$X = \bigcup_{i=1}^{m(s)} U_{1/s,x(i,s)} \,. \tag{2.1}$$

Let $A = \bigcup_{s=1}^{\infty} A_s$. As every A_s is a finite set, the set A is countable. Put $A = \{a_1, a_2, a_3, ...\}$. Note that for each $x \in X$ the sequence $(f_n(x))$, as it lies in the compact metric space Y, has a convergent subsequence. We are now looking for a subsequence of f_n which converges at all $a_i \in A$, and then we will prove that it uniformly converges on X.

Let $g_n^{(1)}$ be a subsequence of f_n which converge at a_1. For the same reason we can find a subsequence $g_n^{(2)}$ of $g_n^{(1)}$ that converges at a_2, and $g_n^{(2)}$ being a subsequence of $g_n^{(1)}$, it converges at a_1 as well. Next, we can find a subsequence $g_n^{(3)}$ of $g_n^{(2)}$ that converges at a_3, and $g_n^{(3)}$ being a subsequence of $g_n^{(2)}$, it converges at a_1 and a_2 as well. By continuing this process we inductively find sequences $g_n^{(h)}$ for each natural h such that $g_n^{(h+1)}$ is a subsequence of $g_n^{(h)}$ for each h, and $g_n^{(h)}$ converges at $a_1, a_2, ..., a_h$. Clearly, every $g_n^{(h)}$ is a subsequence of f, and the problem is that each of those subsequences only converges (a priori) at a *finite* subset of A.

By what a way we can find a subsequence that converges at *all* points in A? The answer is: we take the *diagonal* subsequence defined by $g_n = g_n^{(n)}$. Indeed, g_n is a subsequence "from h on" of $g_n^{(h)}$ for each h in the sense that there exists a strictly increasing map ψ_h from $\{h, h+1, ...\}$ into itself so that $g_n = g_{\psi_h(n)}^{(h)}$ for every $n \geq h$, and therefore on one hand g_n is a subsequence of f_n, on the other g_n converges at a_h for each h.[3] Prove now that g_n is uniformly convergent on X. As Y is a compact, thus complete, metric space, this amounts to prove that g_n is uniformly Cauchy. *Given* $\varepsilon > 0$, let $s = 1, 2, 3, ...$ be so that $\frac{3}{s} < \varepsilon$. As $g_n(a_i)$ converges for all i, in particular $g_n(x(i, s))$ converges for all $i = 1, ..., m_s$. Hence *there exists* $\nu \in \mathbb{N}$ such that when $h, k \geq \nu$ then

$$d\big(g_h(x(i,s)), g_k(x(i,s))\big) < \frac{1}{s} \tag{2.2}$$

for each $i = 1, ..., m_s$. This as for each $i = 1, ..., m_s$ we find ν_i so that (2.2) holds (for that i) for each $h, k \geq \nu_i$. Then, we take $\nu = \max \nu_i$. *Let now* $x \in X$. In view of (2.1)

[3] To see this, note that if $m \geq h$ $g_n^{(m)}$ is a subsequence of $g_n^{(h)}$ thus there exists $\phi_m : \{1, 2, ...\}$ to itself so that $g_n^{(m)} = g_{\phi_m(n)}^{(h)}$. As $g_n^{(m+1)}$ is a subsequence of $g_n^{(m)}$ we find inductively ϕ_m, in particular $g_n^{(m+1)} = g_{\sigma(n)}^{(m)} = g_{\phi_m(\sigma(n))}^{(h)}$ for some strictly increasing σ, hence $\phi_{m+1} = \phi_m \circ \sigma$, and $\phi_{m+1} \geq \phi_m$. Thus, for $n \geq h$, $g_n = g_n^{(n)} = g_{\psi_h(n)}^{(h)}$ with $\psi_h(n) = \phi_n(n)$ and as $\phi_{n+1}(n+1) \geq \phi_n(n+1) > \phi_n(n)$, ψ_h is strictly increasing.

there exists $i = 1, \ldots, m_s$ such that $x \in U_{1/s, x(i,s)}$, so that, by the definition of $U_{\varepsilon,x}$ we have $d\big(g_m(x), g_m(x(i,s))\big) < \frac{1}{s}$ for all m as every g_m is of the form f_n for some n. Thanks to (2.2), it follows that for $h, k \geq \nu$,

$$d(g_k(x), g_h(x)) \leq d\big(g_k(x), g_k(x(i,s))\big) + d\big(g_k(x(i,s)), g_h(x(i,s))\big) + d\big(g_h(x(i,s)), g_h(x)\big)$$

$$< \frac{3}{s} < \varepsilon\,.$$

As ε is an arbitrary positive number, g_n is uniformly Cauchy, thus it uniformly converges. ∎

We cannot apply the previous theorem when $Y = \mathbb{R}^M$ as in such a case Y is not compact. However, if there exists $K > 0$ such that

$$||f_n(x)|| \leq K \quad \forall x \in X \quad \forall n, \tag{2.3}$$

we can consider $f_n : X \to \overline{B(0,K)}$ and as $\overline{B(0,K)}$ is compact we can apply Theorem 2.2 again. When (2.3) holds the function f_n are said to be equibounded as they are bounded by a constant which is independent of n. We thus have the following corollary, which is one of the most usual forms of the Ascoli-Arzelà Theorem.

Corollary 2.3. *If f_n are equicontinuous and equibounded functions from a compact topological space X to $\mathbb{R}^M$, then there exists a subsequence of f_n uniformly convergent on X.* ∎

When X is a metric space with distance d', a typical case in which the functions f_n are equicontinuous is that in which they are *equilipshitzian*, i.e., there exists $K > 0$ so that $d(f_n(x), f_n(y)) \leq K d'(x,y)$ for each $x, y \in X$ and for each n. In other words, they satisfy a Lipschitz condition with a constant independent of n. In fact, in this case it suffices to take $\delta = \frac{\varepsilon}{K}$ in the definition of (uniform) equicontinuity. As a particular case, if f_n are functions defined on an interval in $\mathbb{R}$ with values in $\mathbb{R}$, they are equilipshitzian when they have equibounded derivatives. Indeed, by the mean value Theorem, $|f_n(x) - f_n(y)| \leq \big(\sup |f_n'|\big)|x - y|$.

Exercise 2.1. Prove that Theorem 2.2 (or Corollary 2.3) is no longer valid if $X = \mathbb{R}$.

Exercise 2.2. Prove that if f_n are equibounded and equicontinuous functions from $\mathbb{R}^N$ to $\mathbb{R}^M$ (more generally if they are equibounded on every compact subset of $\mathbb{R}^N$ and equicontinuous), then there exists a subsequence of f_n uniformly convergent on the compact subsets of $\mathbb{R}^N$.

Exercise 2.3. Prove that the conclusion of the previous exercise is still valid if $\mathbb{R}^N$ is replaced by any open subset of $\mathbb{R}^N$.

Exercise 2.4. Find a sequence of equibounded functions from $[0,1]$ to $\mathbb{R}$ which has no subsequence *pointwise* convergent.

3. Curves of Minimum length

The purpose of this section is to prove the following

Theorem 3.1. *Given a closed subset A of $\mathbb{R}^N$ and two points $P, Q \in A$ such that*
a) there exists a continuous curve in A connecting them having finite length,
then there exists a continuous curve in A connecting them having minimum length.

Note that a) in Theorem 3.1 for any $P, Q \in A$, is a condition stronger than arcwise connectedness, in the sense that arcwise connectedness requires that any two points $P, Q \in A$ can be connected by a continuous curve but not necessarily having finite length. In order to clarify the statement in Theorem 3.1, first of all, we recall the definitions concerning the length of a curve. Given a closed interval $[a, b]$ (with $a, b \in \mathbb{R}$, $a < b$), a *partition* of $[a, b]$ is an object of the form $(t_0, t_1, ..., t_n)$ such that $a = t_0 < t_1 < ... < t_n = b$. We denote by $\mathcal{P}_{a,b}$ the set of the partitions of $[a, b]$. A continuous curve in a subset A of $\mathbb{R}^N$ is a continuous function from a closed interval $[a, b]$, $(a < b)$, to A. Given $\Pi = (t_0, t_1, ..., t_n) \in \mathcal{P}_{a,b}$, and a continuous curve from $[a, b]$ to $\mathbb{R}^N$, we denote by $\Lambda_\gamma(\Pi)$ the real number

$$\sum_{i=1}^{n} ||\gamma(t_i) - \gamma(t_{i-1})||$$

and we define the length of γ to be the nonnegative, possibly infinite, value

$$L(\gamma) := \sup_{\Pi \in \mathcal{P}_{a,b}} \Lambda_\gamma(\Pi) .$$

We recall that if γ is piecewise C^1, then we have the formula

$$L(\gamma) = \int_a^b ||\gamma'(t)|| \, dt .$$

We also recall that the length of a curve is invariant up to a reparametrization. In order to clarify this, we recall that given a continuous curve $\gamma : [a, b] \to \mathbb{R}^N$, a reparametrization of it is a curve $\tilde{\gamma} : [c, d] \to \mathbb{R}^N$ defined by $\tilde{\gamma} = \gamma \circ \phi^{-1}$ where ϕ is a continuous bijection from $[a, b]$ onto $[c, d]$ (Note that, in such a case, by a well-known theorem, the inverse ϕ^{-1} is continuous as well). Then the length of a curve amounts to the length of any reparametrization.

We now equip the set $\mathcal{C}_{a,b;N}$ of the continuous curves from a fixed interval $[a, b]$ to $\mathbb{R}^N$ with the norm

$$||\gamma||_\infty = \sup_{t \in [a,b]} ||\gamma(t)||.$$

We recall that the convergence induced by such a norm is the uniform convergence, in other words, $\gamma_n \underset{n\to\infty}{\longrightarrow} \gamma$ in $|| \; ||_\infty$ if and only if $\gamma_n \underset{n\to\infty}{\longrightarrow} \gamma$ uniformly. If we consider $L(\gamma)$

as a function of γ, we realize that L is not continuous, as we can approximate a curve of finite length by a sequence of curves having length tending to infinity. For example, it is easy to see that the curve $(t, 0)$ on $[0, 1]$ in $\mathbb{R}^2$ has length 1 and the approximating curves $\left(t, \frac{1}{\sqrt{n}}\sin(2\pi nt)\right)$ have length tending to infinity. Nevertheless, the intuition suggests that, if we approximate a curve γ in $\|\ \|_\infty$, the length could greatly increase but not greatly decrease, in other words the function L from the set of the continuous curves into $\mathbb{R}\cup\{+\infty\}$ is not continuous, but *lower semicontinuous*. We recall the following definition.

Definition 3.2. *Let F be a function from a topological space X to $\mathbb{R}\cup\{+\infty\}$. We say that F is lower semicontinuous (abbreviated as l.s.c.) at a point $x \in X$ if for every $M \in \mathbb{R}$ such that $M < F(x)$ there exists U neighborhood of x in X such that for every $y \in U$ we have $F(y) > M$. We say that F is lower semicontinuous (on X) if F is lower semicontinuous at each point in X.*

We could give the definition of l.s.c. in the more natural setting of functions with values in $\mathbb{R}$, but we prefer to do this in the setting of functions with values in $\mathbb{R} \cup \{+\infty\}$, as we will study it in the case of the length of a curve that can well assume the value $+\infty$. Moreover, we will study the sup of a family of l.s.c. functions, which can assume the value $+\infty$ even if all the functions take finite values. Note that if the function F only assumes finite values, then the definition of lower semicontinuity can be expressed as: F is l.s.c. at x if for every $\varepsilon > 0$ the exists U neighborhood of x in X such that for every $y \in U$ we have $f(y) > f(x) - \varepsilon$. So, we see the difference with respect to the definition of continuity at x, where we require that in a suitable neighborhood of x we have $f(x) + \varepsilon > f(y) > f(x) - \varepsilon$, in other words, in the definition of semicontinuity we require that in a neighborhood of x the function is not too smaller than at x, but not necessarily not too greater than at x.

If (X, d) is a metric space and F only takes finite values, of course the semicontinuity can be also expressed using ε and δ, i.e., F is l.s.c. at x if for every $\varepsilon > 0$ the exists $\delta > 0$ such that for every $y \in X$ such that $d(x, y) < \delta$ we have $f(y) > f(x) - \varepsilon$. Of course, every continuous function at x is l.s.c. at x, but the converse is not true, for example the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 2$ if $x \neq 0$, $F(0) = 1$, is l.s.c. but not continuous at 0. Note that in the definition of lower semicontinuity we use the order stucture of $\mathbb{R}$ or of $\mathbb{R} \cup \{+\infty\}$, so that such a definition, unlike the definition of continuity, does not make sense for functions with values in an arbitrary topological (or even metric) space.

We recall that if X is a metric space the continuity at x can be expressed in terms of convergences of sequences, namely F is continuous at x if and only if, for every sequence (x_n) in X tending to x, we have $F(x_n) \underset{n\to+\infty}{\longrightarrow} F(x)$. A similar characterization holds for lower semicontinuity, namely (under the hypothesis of Def. 3.2) F is l.s.c. at $x \in X$ if and only if for every sequence (x_n) in X tending to x, we have $\liminf_{n\to+\infty} F(x_n) \geq F(x)$. We omit the proof, which resembles that for the continuity. We only note that the part $\Rightarrow$ is rather simple and does not use the fact that X is a metric space, and the part $\Leftarrow$ is proved by contradiction and would not be valid in the general case of X topological space.

We now are going to prove that the sup of l.s.c. functions (at a point) is l.s.c. (at that point). Note that in general the continuity does not have the same property, e.g., the functions $1 - x^n$ from $[0, 1]$ to $\mathbb{R}$ are continuous, but the sup of them, when n varies on $1, 2, 3, ...$, is the function f defined by $f(x) = 1$ if $x < 1$, $f(1) = 0$, which is discontinuous at 1.

Lemma 3.3. *Suppose f_i, $i \in I$ are functions from a topological space X with values in $\mathbb{R} \cup \{+\infty\}$, and $f = \sup_{i \in I} f_i$. If f_i are l.s.c. at $x \in X$ then f is l.s.c. at x.*

Proof. Let $M \in \mathbb{R}$ be such that $M < f(x)$. As $f(x) = \sup_{i \in I} f_i(x)$, there exists $i \in I$ such that $f_i(x) > M$, and as f_i is l.s.c. at x there exists U neighborhood of x such that for every $y \in U$ we have $f_i(y) > M$. Hence, for every $y \in U$ we have $f(y) \geq f_i(y) > M$, and as M is an arbitrary number less than $f(x)$, f is l.s.c. at x. ■

Corollary 3.4. *Let $a, b \in \mathbb{R}$, $a < b$. Then the function L defined on $\left(\mathcal{C}_{a,b;N}, || \, ||_\infty\right)$ by $\gamma \mapsto L_\gamma$ is l.s.c.*

Proof. In view of Lemma 3.3, it suffices to prove that, for each $\Pi \in \mathcal{P}_{a,b}$, the map $\gamma \mapsto \Lambda_\gamma(\Pi)$ from $\mathcal{C}_{a,b;N}$ to $\mathbb{R}$, is continuous. Let $\Pi = (t_0, t_1, ...t_n)$. Since, clearly, the map $\gamma \mapsto \gamma(t)$ from $\mathcal{C}_{a,b;N}$ to $\mathbb{R}^N$ is continuous for every $t \in [a, b]$, hence so is the map $\gamma \mapsto \gamma(t_i) - \gamma(t_{i-1})$ for $i = 1, ..., n$, as the difference of continuous functions. Hence, the map $\gamma \mapsto \Lambda_\gamma(\Pi)$ is continuous as the sum of the composition of the norm function, which is continuous from $\mathbb{R}^N$ to $\mathbb{R}$, with continuous functions. ■

We now sketch the plan of the proof of Theorem 3.1. It is possible to prove that a l.s.c. function from a (nonempty) compact topological space to $\mathbb{R}$ has a minimum. Now, the space $\mathcal{C}_{a,b;N}$ is not compact, but we can restrict the l.s.c. function L to a suitable subset X of $\mathcal{C}_{a,b;N}$. As X, being a subset of the metric space $\mathcal{C}_{a,b;N}$ is a metric space as well, in order to see whether X is a compact, we have to check whether every sequence of functions in X has a subsequence convergent to an element of X with respect to the norm $|| \, ||_\infty$, that is, uniformly. Thus, the idea constists in finding a suitable X composed by equibounded and equicontinuous functions, so that we can apply the Ascoli-Arzelà Theorem. First, we can consider the space of the curves in $\mathcal{C}_{a,b;N}$ having length less than or equal to a fixed real number k. These curves are not necessarily equicontinuous, but we could reparametrize them by arcleght, so that, as easily verfied, they are Lipshitzian with a Lipshitz constant equal to k. The problem is that not all curves can be reparametrized by arclength, for example it suffices to consider a curve which is constant on some interval. So, in the following we will perform a slight modification of the above idea.

Given $\gamma \in \mathcal{C}_{a,b;N}$ with $L(\gamma) < +\infty$, and $c, d \in [a, b]$, $c \leq d$, we put $L_{c,d}(\gamma) = L(\gamma|_{[c,d]})$, where of course, $\gamma|_{[c,d]}$ denote the restriction of γ to the interval $[c, d]$, with the convention $L_{c,d}(\gamma) = 0$ if $c = d$. It is well known that, if $a \leq c \leq d \leq u \leq b$, then $L_{c,u}(\gamma) = L_{c,d}(\gamma) + L_{d,u}(\gamma)$. We now consider the arclength function $\tilde{\phi} : [a, b] \to \mathbb{R}$ defined by

$\tilde{\phi}(t) = L_{a,t}(\gamma)$. We easily see that $\tilde{\phi}$ is increasing, but not necessarily strictly increasing. We now prove:

Lemma 3.5. *$\tilde{\phi}$ is continuous.*

Proof. We first prove that if $\bar{t} > a$ then $\tilde{\phi}$ is continuous at $\bar{t}$ on the left. Let $\varepsilon > 0$. By the definition of $L_{a,t}(\gamma)$, there exists $\Pi \in \mathcal{P}(a, \bar{t})$ such that $\Lambda_\gamma(\Pi) > \tilde{\phi}(\bar{t}) - \frac{\varepsilon}{2}$. We write $\Pi = (a = t_0, t_1, ..., t_n = \bar{t})$. Let

$$t \in]t_{n-1}, t_n[.$$

Clearly,

$$\Lambda_\gamma(t_0, t_1, ..., t_{n-1}, t, t_n) \geq \Lambda_\gamma(\Pi) > \tilde{\phi}(\bar{t}) - \frac{\varepsilon}{2},$$

$$\Lambda_\gamma(t_0, t_1, ..., t_{n-1}, t, t_n) = \Lambda_\gamma(t_0, t_1, ..., t_{n-1}, t) + ||\gamma(\bar{t}) - \gamma(t)||,$$

hence

$$\tilde{\phi}(\bar{t}) \geq \tilde{\phi}(t) \geq \Lambda_\gamma(t_0, t_1, ..., t_{n-1}, t) = \Lambda_\gamma(t_0, t_1, ..., t_{n-1}, t, t_n) - ||\gamma(\bar{t}) - \gamma(t)||$$

$$> \tilde{\phi}(\bar{t}) - \frac{\varepsilon}{2} - ||\gamma(\bar{t}) - \gamma(t)||.$$

Now, as γ is continuous, there exists $\tilde{t} < \bar{t}$ such that, if $\tilde{t} < t < \bar{t}$ then $||\gamma(\bar{t}) - \gamma(t)|| < \frac{\varepsilon}{2}$. Hence, if $\max\{\tilde{t}, t_{n-1}\} < t < \bar{t}$, then

$$\tilde{\phi}(\bar{t}) \geq \tilde{\phi}(t) > \tilde{\phi}(\bar{t}) - \varepsilon$$

and $\tilde{\phi}$ is continuous at $\bar{t}$ on the left. We now prove that $\tilde{\phi}$ is continuous on the right at any point $\bar{t} < b$. The proof is similar noting that

$$\tilde{\phi}(t) = L(\gamma) - L_{t,b}(\gamma) \quad \forall t \in [a, b]. \tag{3.1}$$

Let $\varepsilon > 0$. We find $\Pi \in \mathcal{P}_{\bar{t},b}$ so that

$$\Lambda_\gamma(\Pi) > L_{\bar{t},b}(\gamma) - \frac{\varepsilon}{2}.$$

We write $\Pi = (\bar{t} = t_0, t_1, ..., t_n = b)$. Let

$$t \in]t_0, t_1[.$$

Clearly,

$$\Lambda_\gamma(t, t_1, ..., t_n) = \Lambda_\gamma(t_0, t, t_1, ..., t_n) - ||\gamma(t) - \gamma(\bar{t})|| \geq \Lambda_\gamma(\Pi) - ||\gamma(t) - \gamma(\bar{t})||.$$

We find $\tilde{t} > \bar{t}$ such that, if $\tilde{t} > t > \bar{t}$ then $||\gamma(\bar{t}) - \gamma(t)|| < \frac{\varepsilon}{2}$. We conclude like before that, if $\bar{t} < t < \min\{\tilde{t}, t_1\}$, then

$$L_{t,b}(\gamma) \geq \Lambda_\gamma(t, t_1, ..., t_n) > L_{\bar{t},b}(\gamma) - \varepsilon$$

so that, using (3.1)

$$\tilde{\phi}(\bar{t}) \leq \tilde{\phi}(t) < \tilde{\phi}(\bar{t}) + \varepsilon$$

and $\tilde{\phi}$ is continuous at $\bar{t}$ on the right. ∎

We now would like to reparametrize γ using the arcleght ϕ, but as previously observed, this is not a strictly increasing function, so we need a modification of it. First we need the following remark.

Remark 3.6. If $\gamma \in \mathcal{C}_{a,b,N}$ and $c, d \in [a, b]$, then $||\gamma(c) - \gamma(d)|| \leq L(\gamma)$. Indeed, if $c = d$ this is trivial, if not we can for example suppose $c < d$. Then one of the following $(a, c, d, b), (a, c, d), (c, d, b), (c, d)$ is in $\mathcal{P}_{a,b}$, depending of what of the inequalities $a \leq c$, $d \leq b$ are strict. Let Π be such an element of $\mathcal{P}_{a,b}$. Then, $L(\gamma) \geq \Lambda_\gamma(\Pi) \geq ||\gamma(c) - \gamma(d)||$.
∎

Since the length of a curve in invariant up to a reparametrization we can and do assume that

$$[a, b] = [0, 1].$$

We now want to reparametrize in a Lipshitzian way a continuous curve from $[0, 1]$ to $\mathbb{R}^N$ of finite length. We modify the arclength function in the following way. Let

$$\phi(t) = \frac{L_{0,t}(\gamma) + t}{L(\gamma) + 1}.$$

In such a definition, we add t to $L_{0,t}(\gamma)$ in order to have a *strictly* increasing function, and divide by $L(\gamma) + 1$ in order that ϕ map $[0, 1]$ onto $[0, 1]$. We easily verify that in fact ϕ is a continuous strictly increasing function from $[0, 1]$ onto itself, so that the curve $\tilde{\gamma}$ defined by

$$\tilde{\gamma} = \gamma \circ \phi^{-1}$$

is a reparametrization of γ. Note now that, if $0 \leq \tau_1 < \tau_2 \leq 1$ then

$$\begin{aligned} L_{\tau_1,\tau_2}(\gamma) = L_{0,\tau_2}(\gamma) - L_{0,\tau_1}(\gamma) &< \frac{L_{0,\tau_2}(\gamma) - L_{0,\tau_1}(\gamma) + \tau_2 - \tau_1}{L(\gamma) + 1}(L(\gamma) + 1) \\ &= \big(\phi(\tau_2) - \phi(\tau_1)\big)(L(\gamma) + 1) \end{aligned} \tag{3.2}$$

so that *if* $0 \leq t_1 < t_2 \leq 1$, using (3.2) with $\tau_1 = \phi^{-1}(t_1)$, $\tau_2 = \phi^{-1}(t_2)$,

$$||\tilde{\gamma}(t_2) - \tilde{\gamma}(t_1)|| \leq L_{\phi^{-1}(t_1),\phi^{-1}(t_2)}(\gamma) \leq (L(\gamma) + 1)(t_2 - t_1),$$

$$||\tilde{\gamma}(t_2) - \tilde{\gamma}(t_1)|| \leq (L(\gamma) + 1)|t_2 - t_1| \tag{3.3}$$

where in the first inequality we have used Remark 3.6 with $\gamma|_{[\phi^{-1}(t_1),\phi^{-1}(t_2)]}$ in place of γ. However, (3.3) holds for every $t_1, t_2 \in [0,1]$, as, if $t_1 > t_2$, we obtain (3.3), by changing t_1 with t_2, and (3.3) is obvious if $t_1 = t_2$. In conclusion, $\tilde{\gamma}$ is Lipshitzian with constant $L(\gamma) + 1$.

Proof of Theorem 3.1. Let $\hat{\gamma}$ be a continuous curve in A, defined on $[0,1]$ having finite length $\overline{L}$ with $\hat{\gamma}(0) = P$, $\hat{\gamma}(1) = Q$. Let

$$X = \left\{\gamma \in \mathcal{C}_{0,1;N} : \gamma(t) \in A \quad \forall t \in [0,1],\ \gamma(0) = P, \gamma(1) = Q,\ L(\gamma) \leq \overline{L}\right\}.$$

Since $\hat{\gamma} \in X$, then $X \neq \emptyset$. Let $\gamma_n \in X$ be such that

$$L(\gamma_n) \underset{n \to +\infty}{\longrightarrow} \inf_{\gamma \in X} L(\gamma). \tag{3.4}$$

Let $\tilde{\gamma}_n$ be a reparametrization of γ_n obtained as above, in particular using a function ϕ (which of course usually depends on n) strictly increasing so that $\tilde{\gamma}_n(0) = \gamma_n(0) = P$, $\tilde{\gamma}_n(1) = \gamma_n(1) = Q$, and $\tilde{\gamma}_n$ Lipshitzian with constant $L(\gamma_n) + 1$. Since $L(\gamma_n) \leq \overline{L}$, we easily see that all $\tilde{\gamma}_n$ are Lipshitzian with constant $\overline{L} + 1$, hence $\tilde{\gamma}_n$ are equilipshitzian. Also, since $L(\tilde{\gamma}_n) = L(\gamma_n)$, $\tilde{\gamma}_n$ being a reparametrization of γ_n, we easily see that $\tilde{\gamma}_n \in X$. Moreover, in view of Remark 3.6, for every $t \in [0,1]$, we have

$$||\tilde{\gamma}_n(t)|| \leq ||\tilde{\gamma}_n(t) - \tilde{\gamma}_n(0)|| + ||\tilde{\gamma}_n(0)|| \leq L(\tilde{\gamma}_n) + ||P|| \leq \overline{L} + ||P||$$

so that $\tilde{\gamma}_n$ are equibounded. We can thus use the Ascoli-Arzelà Theorem and deduce that a suitable subsequence $\tilde{\gamma}_{n_k}$ of $\tilde{\gamma}_n$ converges, for $k \to \infty$, to some $\tilde{\gamma} \in \mathcal{C}_{0,1;N}$ with respect to $||\ ||_\infty$, that is, uniformly. We now see that $\tilde{\gamma} \in X$. In fact, as $\tilde{\gamma}_{n_k}(t) \underset{k \to +\infty}{\longrightarrow} \tilde{\gamma}(t)$ for each $t \in [0,1]$, we have $\tilde{\gamma}(t) \in A$ for each $t \in [0,1]$ (recall that A is assumed to be closed). For the same reason, $\tilde{\gamma}(0) = P$, $\tilde{\gamma}(1) = Q$. Finally, since L is l.s.c. on $\mathcal{C}_{0,1;N}$, we have

$$L(\tilde{\gamma}) \leq \liminf_{k \to +\infty} L(\tilde{\gamma}_{n_k}) = \liminf_{k \to +\infty} L(\gamma_{n_k}) \leq \overline{L}, \tag{3.5}$$

and $\tilde{\gamma} \in X$ as claimed. Using (3.4) and (3.5) we get $L(\tilde{\gamma}) \leq \inf_{\gamma \in X} L(\gamma)$ (thus $L(\tilde{\gamma}) = \min_{\gamma \in X} L(\gamma)$). It remains to prove that $L(\tilde{\gamma}) \leq L(\gamma)$ when γ is a continuous curve in A connecting P and Q, but which is not an element of X. This is obvious, as in such a case $L(\gamma) > \overline{L} \geq L(\tilde{\gamma})$. ■

The argument of Theorem 3.1 can be used to prove the following general theorem, which resembles the well-known Weierstrass Theorem on extrema of continuous functions defined on a compact set.

Theorem 3.7. *Let f be a l.s.c. function from a (nonempty) metric space X to $\mathbb{R}\cup\{+\infty\}$, not identically $+\infty$. Suppose X is compact, or more generally*
a) the set $\{x \in X : f(x) \le L\}$ is compact for each real L.
Then f has a minimum on X.

Proof. Suppose a) holds. Let $\overline{x} \in X$ be so that $L := f(\overline{x}) < +\infty$. Let

$$Y = \{x \in X : f(x) \le L\}.$$

Let $x_n \in Y$ be so that $f(x_n) \underset{n\to+\infty}{\longrightarrow} \inf_{x\in Y} f(x)$. Since Y is compact by our assumption, there exists a subsequence x_{n_k} of x_n and $\tilde{x} \in Y$ such that $x_{n_k} \underset{k\to\infty}{\longrightarrow} \tilde{x}$. Then, as f is l.s.c.,

$$f(\tilde{x}) \le \liminf_{k\to\infty} f(x_{n_k}) = \inf_{x\in Y} f(x).$$

It follows that $f(\tilde{x}) \le f(x)$ for each $x \in Y$, in particular $f(\tilde{x}) \le f(\overline{x})$ but also $f(\tilde{x}) \le f(x)$ for each $x \in X \setminus Y$, as in such a case $f(\tilde{x}) \le f(\overline{x}) < f(x)$. In conclusion, f takes its minimum at $\tilde{x}$. Note that if X is compact, then a) holds since, as a consequence of the lower semicontinuity of f, the set $Z := \{x \in X : f(x) \le L\}$ is closed (if $x \notin Z$, then $f(x) > L$, hence there exists U neighborhood of x such that $f(y) > L$ for each $y \in U$, and therefore the complement of Z is open) in the compact set X, hence it is compact. ∎

The fact that a continuous function on $\mathbb{R}$ with values in $\mathbb{R}$ which tends to $+\infty$ both at $-\infty$ and at $+\infty$ has a minimum can be seen as a particular case of Theorem 3.7, as in such a case a) in Theorem 3.7 holds. In general, a function f satisfying a) in Theorem 3.7, is said to be coercive. The coerciveness condition is important in many problems in calculus of variations as usually the domain of a functional is not compact, but in many examples the functional is coercive.

Exercise 3.1. Prove that, if A is a nonempty *open* subset of $\mathbb{R}^N$, then any two points in A are connected by a continuous curve of minimum length if and only if A is convex. It follows that the hypothesis that A is closed cannot be removed in Theorem 3.1.

4. Geodesics on surfaces

In the previous section, we proved Theorem 3.1. In the following, we will call geodesic in a subset A of $\mathbb{R}^N$ a curve satisfying a) of Theorem 3.1 for some $P, Q \in A$. Note that in differential geometry the term *geodesic* is used in a slightly different sense. The geodesic connecting two given points is not unique as for example any (verse-preserving) reparametrization of it satisfies the same property. Further, in some cases, we can have essentially different geodesics (i.e., having a different image) connecting two given points. For example, if A is a sphere in $\mathbb{R}^3$, and the two points are the north and the south pole, then every meridian is such a geodesic. If the closed subset A of $\mathbb{R}^N$ has no further properties we cannot in general find a *regular* (say C^1) geodesic connecting two given points. An example is when $N = 2$ and A is the complement of an open square, and the two points lie on different edges of the square. This fact is highly intuitive and will not prove this fact in detail.

Suppose now we have a surface S in $\mathbb{R}^3$ and we will assume it is compact, connected and, for simplicity, of class C^∞. In this case it is possible to prove that given two points there exists a regular geodesic in S connecting them, that can be parametrized by arclength. We recall that a C^1 curve γ is parametrized by arclength if the vector $||\gamma'||$ is identically 1. The purpose of this section is to give a characterization of (sufficiently smooth) geodesics on S. We recall that a (nonempty) subset S of $\mathbb{R}^3$ is said to be a surface if

a) For every $\overline{P} \in S$ there exists U open neighborhood of $\overline{P}$ and $g : U \to \mathbb{R}$ of class C^∞ such that $\mathrm{grad} g \neq 0$ *on U and $S \cap U = \Big\{P \in U : g(P) = 0\Big\}$,*

or equivalently

b) For every $\overline{P} \in S$ there exists U open neighborhood of $\overline{P}$ and V open set in $\mathbb{R}^2$, and $\phi : V \to \mathbb{R}^3$ of class C^∞ such that the rank of the Jacobian matrix of ϕ equals 2 on V and $S \cap U = \phi(V)$,

or also

c) S can be locally represented as the graph of a C^∞ function ψ of the form $z = \psi(x, y)$ or $y = \psi(z, x)$ or $x = \psi(y, z)$, more precisely, for every $\overline{P} \in S$ there exists U open neighborhood of $\overline{P}$ and V open in $\mathbb{R}^2$, and $\psi : V \to \mathbb{R}$ of class C^∞ such that one of the following holds
c_1) $S \cap U = \big\{(x, y, z) : (x, y) \in V, z = \psi(x, y)\big\}$,
c_2) $S \cap U = \big\{(x, y, z) : (z, x) \in V, y = \psi(z, x)\big\}$,
c_3) $S \cap U = \big\{(x, y, z) : (y, z) \in V, x = \psi(y, z)\big\}$.

We shortly recall how it can be proved that a) b) and c) are equivalent: If a) holds, then one of the derivatives $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$, $\frac{\partial g}{\partial z}$ is different from 0 at $\overline{P}$, thus in a neighborhood of $\overline{P}$. Suppose for example $\frac{\partial g}{\partial z} \neq 0$ at $\overline{P}$. Then, as a consequence of the implicit function Theorem, we can represent S as in c_1), and c) holds. If b) holds, then one of the three submatrices of 2×2 of the jacobian matrix of $\phi = (\phi_1, \phi_2, \phi_3)$ is nonsingular at $\overline{Q}$ with $\phi(\overline{Q}) = \overline{P}$. Suppose for example it is the matrix relative to (ϕ_1, ϕ_2). Then, writing ϕ as $\phi(u, v)$, by the inverse function Theorem we can express locally (u, v) as a C^∞ function h of (x, y). Thus, letting $\psi = \phi_3 \circ h$, we can represent S as in c_1). On the other hand if c), for example c_1), holds, then we get a) putting $g(x, y, z) = z - \psi(x, y)$, and b) putting $\phi(u, v) = \big(u, v, \psi(u, v)\big)$. Now, in c), for example c_1), we can assume

$$\overline{P} = 0, \quad \frac{\partial \psi}{\partial x}(0, 0) = \frac{\partial \psi}{\partial y}(0, 0) = 0 \tag{4.1}$$

in the sense that if (4.1) does not hold, then the image of S via a suitable affine isometry satisfies (4.1). Once we realize that the second condition in (4.1) means that the tangent plane Π to S at $\overline{P}$ is the plane Π' of equation $z = 0$, this can be seen, observing that there exists an affine isometry that carries $\overline{P}$ into 0 and Π into Π'. By this point of view, the assumption (4.1) is valid when we treat properties invariant whith respect to affine isometries. Finally, we recall, that given a C^2 curve γ in $\mathbb{R}^3$, parametrized by arclength, then the principal normal to γ at the point $\gamma(t)$ is $\gamma''(t)$.

Remark 4.1. We explicitly note that if S is a compact and connected surface in $\mathbb{R}^3$ of class C^∞, then any two points Q and P can be connected by a piecewise C^1, hence of finite length, curve in S, thus S satisfies the hypothesis of Theorem 3.1. The argument of the proof resembles that used for proving that, in an open connected set U in $\mathbb{R}^N$, any two points can be connected by a polygonal that remains in U. Fixing the point Q, let

$$A = \left\{\overline{P} \in S : Q \text{ and } \overline{P} \quad \text{are connected by a piecewise } C^1 \text{ curve in } S\right\}.$$

Then $Q \in A$ so that $A \neq \emptyset$. We will see that A is closed and open, hence by the assumption that S is connected, $A = S$. Let $\overline{P} \in S$. Let us represent S using c), for example c_1, with $\overline{P} = (\overline{x}, \overline{y}, \psi(\overline{x}, \overline{y}))$. Consider an open ball B in $\mathbb{R}^2$ with center at $(\overline{x}, \overline{y})$ contained in V and let

$$W := \left\{(x, y, z) : (x, y) \in B, z = \psi(x, y)\right\}.$$

Since $W = S \cap U \cap \pi^{-1}(B)$, where the continuous function $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $\pi(x, y, z) = (x, y)$, then W is open in S. Moreover, any $P \in W$ is connected to $\overline{P}$ by a C^1 curve γ in S. In fact, let $P = (x, y, \psi(x, y))$, let $\tilde{\gamma}$ be the segment line connecting $(\overline{x}, \overline{y})$ to (x, y) (which lies in B). Then, it suffices to take γ defined by $\gamma(t) = (\tilde{\gamma}(t), \psi(\tilde{\gamma}(t))$. It easily follows that, if $\overline{P} \in A$ then $W \subseteq A$, while if $\overline{P} \in S \setminus A$, then $W \subseteq S \setminus A$. It follows that in fact both A and $S \setminus A$ are open in S, hence A is closed and open in S. ∎

Theorem 4.2. *Suppose S is a compact and connected surface in $\mathbb{R}^3$ of class C^∞, and let $\gamma : [a, b] \to S$ be a geodesic of class C^2 in S connecting two given points $P, Q \in S$, parametrized by arclength. Then, for any $\overline{t} \in]a, b[$, the principal normal to γ at $\gamma(\overline{t})$ is normal to the surface at $\gamma(\overline{t})$.*

Proof. Put $\overline{P} := \gamma(\overline{t})$. Then we can assume that c_1) holds, and also, (4.1) holds, as the properties we use are invariant with respect to an affine isometry. As $\gamma(\overline{t}) \in U$, by a continuity argument, there exist c, d with $a \leq c < \overline{t} < d \leq b$ such that $\gamma(t) \in U$ for each $t \in [c, d]$. Then $\gamma|_{[c,d]}$ minimizes the length of the curves connecting $\gamma(c)$ to $\gamma(d)$ in $S \cap U$ as if there would be a curve η connecting $\gamma(c)$ to $\gamma(d)$ in $S \cap U$, with $L(\eta) < L(\gamma|_{[c,d]})$, then, replacing in γ the piece from c to d by η we would obtain a curve connecting P, Q in S of length less than $L(\gamma)$, a contradiction. Since any $\eta : [c, d] \to S \cap U$ has the form

$$\eta(t) = \Big(y_1(t), y_2(t), \psi(y_1(t), y_2(t))\Big),$$

it follows that (γ_1, γ_2) minimizes the integral

$$\int_c^d \sqrt{y_1'(t)^2 + y_2'(t)^2 + \frac{d}{dt}\big(\psi(y(t))\big)^2} \tag{4.2}$$

among the C^1 $y : [c,d] \to V$ satisfying the conditions

$$y(c) = (\gamma_1(c), \gamma_2(c)), \quad y(d) = (\gamma_1(d), \gamma_2(d)).$$

Thus, $\overline{y} = (\gamma_1, \gamma_2)$ has to satisfy the Euler equation

$$\frac{d}{dt}\left(\frac{y_1'(t) + y_1'(t)\left(\frac{\partial\psi}{\partial y_1}(y(t))\right)^2 + y_2'(t)\,\frac{\partial\psi}{\partial y_1}(y(t))\,\frac{\partial\psi}{\partial y_2}(y(t))}{\sqrt{y_1'(t)^2 + y_2'(t)^2 + \frac{d}{dt}\big(\psi(y(t))\big)^2}}\right)$$

$$= \frac{\partial}{\partial y_1}\left(\sqrt{y_1'(t)^2 + y_2'(t)^2 + \frac{d}{dt}\big(\psi(y(t))\big)^2}\right)$$

for every $t \in [c,d]$, in particular for $t = \overline{t}$. Now, by our assumption, for $y = \overline{y}$, the denominator is constant in t, as it represents $||\gamma'(t)||$ and γ is parametrized by arclength. Hence, the derivative of the numerator is 0, but for $t = \overline{t}$, this amounts to $\overline{y}_1''(\overline{t}) = 0$, as

$$\frac{\partial\psi}{\partial y_1}(\overline{y}(\overline{t})) = \frac{\partial\psi}{\partial y_2}(\overline{y}(\overline{t})) = 0 \tag{4.3}$$

and $\overline{P} = \gamma(\overline{t}) = (0,0,0)$. Similarly, we get $\overline{y}_2''(\overline{t}) = 0$, thus $\gamma''(\overline{t})$ is a multiple of the vector $(0,0,1)$ which is, in view of (4.1), normal to S at $\overline{P}$. ∎

Remark 4.3. In the previous proof, we have used the representation of S in c_1), but such a representation is only valid in a neighborhood of $\overline{P}$, and as there is no reason that the curve γ lies in such a neighborhood, we have to work on the restriction of γ to a suitable neighborhood of $\overline{t}$. This is the reason for which we have not considered the integral on all of the interval $[a,b]$, but we have restricted it to $[c,d]$. We also remark that the considerations in previous proof are only valid at $\overline{t}$ as at the other points (4.3) does not (necessarily) hold, but once t is given we can use a suitable affine isometry (depending on t) for which (4.3) holds at t. ∎

5. Absolutely Continuous Functions

I recall that a function $u : [a,b] \to \mathbb{R}^N$ is absolutely continuous (shortly AC) if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that

$$\Big(a \le a_1 < b_1 \le a_2 < b_2 \cdots \le a_n < b_n,\ \sum_{i=1}^n (b_i - a_i)\ < \delta\Big) \Rightarrow \sum_{i=1}^n ||u(b_i) - u(a_i)||\ < \varepsilon\,.$$

I will denote the set of the AC function from $[a,b]$ to $\mathbb{R}^N$ by $AC([a,b],\mathbb{R}^N)$ or simply by AC when $[a,b]$ and N are clear from the context. It is easy to see that u is AC if and only if u_i is AC for every $i=1,...,N$. Recall that in such situations we can use the inequalities $\max\limits_{i=1,\ldots,N}|u_i| \le ||u|| \le \sum\limits_{i=1,\ldots,N}|u_i|$. I also recall that a Lipshitzian function is AC and that every AC function has a derivative a.e. and that if u is AC then $u' \in L^1([a,b],\mathbb{R}^N)$, and the following form of the fundamental theorem of integral calculus holds:

$$u(x) = u(a) + \int_a^x u'(t)\,dt \quad \forall\, x \in [a,b]\,.$$

Moreover, a sort of converse holds: If $f \in L^1([a,b],\mathbb{R}^N)$ and $u:[a,b]\to\mathbb{R}^N$ satisfies

$$u(x) = u(a) + \int_a^x f(t)\,dt \quad \forall\, x \in [a,b]\,,$$

then u is AC on $[a,b]$. I denote by $Lip([a,b],\mathbb{R}^N)$ the set of the Lipshitzian functions from $[a,b]$ to $\mathbb{R}^N$ and by $Lip_0([a,b],\mathbb{R}^N)$ the set $\{v \in Lip_0([a,b],\mathbb{R}^N) : v(a)=v(b)=0\}$. I recall that a measurable function $f:[a,b]\to\mathbb{R}^N$ is by definition in $L^1([a,b],\mathbb{R}^N)$ if $\int\limits_{[a,b]}||f|| < +\infty$. As for the case of AC functions the notations $Lip([a,b],\mathbb{R}^N)$, $L^1([a,b],\mathbb{R}^N)$ and similar can be abbreviated to Lip, L^1 and similar. Moreover, $f \in Lip$ if and only if $f_i \in Lip$ for every $i=1,...,N$, $f\in L^1$ if and only if $f_i \in L^1$ for every $i=1,...,N$.

I recall that if $f \in L^1([a,b],\mathbb{R}^N)$ then for every $\varepsilon>0$ there exists $\delta>0$ such that if $A\subseteq[a,b]$ and $\mu(A)<\delta$ then $\int_A ||f(x)||dx < \varepsilon$, μ denoting Lebesgue measure. A set of functions $\mathcal{F}=\{f_{(i)} : i \in I\}$ in $L^1([a,b],\mathbb{R}^N)$ is said to be equiintegrable if for every $\varepsilon>0$ there exists $\delta>0$ such that if $A\subseteq[a,b]$ and $\mu(A)<\delta$ then $\int_A ||f_{(i)}(x)||dx<\varepsilon$ for every $i\in I$. Note that I use the notation $f_{(i)}$ instead of f_i in order to distinguish the function $f_{(i)}$ from the i-th component of f. Note also that in such a case the set $\mathcal{F}$ is bounded in L^1, i.e., $||f_{(i)}||_{L^1}\le K$ for some K independent of $i\in I$. To see this, take δ corresponding to $\varepsilon=1$ in the previous definition, and note that $[a,b]\subseteq\bigcup\limits_{i=1}^m A_i$ with $\mu(A_j)<\delta$, for example $A_j=[a+\frac{j-1}{m}(b-a), a+\frac{j}{m}(b-a)]$, with m so large that $\frac{b-a}{m}<\delta$. Then, clearly,

$$||f_{(i)}||_{L^1} = \int_a^b ||f_{(i)}(x)||\,dx \le \sum_{j=1}^m \int_{A_j} ||f_{(i)}(x)||\,dx \le m\,.$$

Consequently, if $\mathcal{F}$ is equiintegrable and A is a measurable subset of $[a,b]$, then we have

$$\Big|\Big|\int_A f_{(i)}\Big|\Big| \le \int_A ||f_{(i)}|| \le ||f_{(i)}||_{L^1} \le m \tag{5.1}$$

We have the following lemma.

Lemma 5.1. *If $f_{(n)}$ is a sequence of AC functions from $[a,b]$ to $\mathbb{R}^N$ whose derivatives are equiintegrable, and such that the sequence $(f_{(n)}(a))$ is bounded, then there exists a subsequence of $f_{(n)}$ that uniformly converges to an AC function.*

Proof. Let ε and δ be as in the definition of equiintegrability. Let $a \leq a_1 < b_1 \leq a_2 < b_2 \cdots \leq a_n < b_n$ be given with $\sum_{i=1}^{n}(b_i - a_i) < \delta$. Then

$$\sum_{i=1}^{n} \left|\left|f_{(n)}(b_i) - f_{(n)}(a_i)\right|\right| = \sum_{i=1}^{n} \left|\left| \int_{a_i}^{b_i} f'_{(n)}(t)dt \right|\right| \leq \sum_{i=1}^{n} \int_{a_i}^{b_i} \left|\left|f'_{(n)}(t)\right|\right| dt = \int_A \left|\left|f'_{(n)}(t)\right|\right| dt < \varepsilon \tag{5.2}$$

where $A = \bigcup_{i=1}^{n} [a_i, b_i]$, as $\mu(A) = \sum_{i=1}^{n} \mu([a_i, b_i]) < \delta$. From (5.2) it follows that f_n are in particular equicontinuous, and as they are equibounded at a by hypothesis, we easily see that they are equibounded. By the Ascoli-Arzelà Theorem $f_{(n)}$ has a subsequence that uniformly converges to a function f, which is AC, as we can pass to the limit in (5.2), obtaining

$$\sum_{i=1}^{n} ||f_{(n)}(b_i) - f_{(n)}(a_i)|| \leq \varepsilon . \quad \blacksquare$$

Lemma 5.2. *A sequence $f_{(n)}$ of equiintegrable functions from $[a,b]$ to $\mathbb{R}^N$ has a subsequence weakly convergent in L^1.*

Proof. By the definition of weak convergence we have to prove that for some subsequence $f_{(n_k)}$ of $f_{(n)}$, and $f \in L^1$ we have

$$\int_{[a,b]} f_{(n_k)} \cdot g \underset{k\to+\infty}{\longrightarrow} \int_{[a,b]} f \cdot g \tag{5.3}$$

for every $g \in L^\infty$. Clearly, we can and do assume $N = 1$. Also, by decomposing a function into its positive and negative part, we can assume $f_{(n)} \geq 0$. We first prove that for some subsequence $f_{(n_k)}$ of $f_{(n)}$, we have

$$\exists \quad \lim_{k\to\infty} \int_A f_{(n_k)} \in \mathbb{R} \tag{5.4}$$

for every measurable set $A \subseteq [a,b]$. Note that due to (5.1) we can find $f_{(n_k)}$ such that (5.4) holds for a single set A, thus using a diagonal process, for countably many sets A_i. So, we can and do assume that (5.4) holds when $A =]\alpha, \beta[$ with $a \leq \alpha < \beta \leq b$, and $\alpha, \beta \in \mathbb{Q}$, as such a set of intervals is countable. We will now prove that for this subsequence $f_{(n_k)}$, (5.4) holds for every measurable set $A \subseteq [a,b]$, by proving that the sequence $\int_A f_{(n_k)}$ is a Cauchy sequence. We can and do assume $A \subseteq]a,b[$. If A is open (in $]a,b[$ or equivalently in $\mathbb{R}$), then, either by topological considerations or using the Vitali covering Theorem, we have that, given $\varepsilon > 0$, and δ *corresponding to* $\frac{\varepsilon}{3}$ *in the definition of equiintegrability*, there exist finitely many mutually disjoint intervals $]\alpha_i, \beta_i[$, $i = 1, ..., m$ with $\alpha_i, \beta_i \in \mathbb{Q}$ such that $\bigcup_{i=1}^{m}]\alpha_i, \beta_i[\subseteq A$ and

$$\mu(V) < \delta,\ V := \Big(A \setminus (\bigcup_{i=1}^{m}]\alpha_i, \beta_i[)\Big)\,.$$

Hence, setting $B = \bigcup_{i=1}^{m}]\alpha_i, \beta_i[$, for every indices k, k' we have

$$\Big|\int_A f_{(n_k)} - \int_A f_{(n_{k'})}\Big| \leq$$

$$\Big|\int_A f_{(n_k)} - \int_B f_{(n_k)}\Big| + \Big|\int_B f_{(n_k)} - \int_B f_{(n_{k'})}\Big| + \Big|\int_B f_{(n_{k'})} - \int_A f_{(n_{k'})}\Big|\,.$$

The first term in the sum amounts to

$$\Big|\int_V f_{(n_k)}\Big| \leq \int_V \Big|f_{(n_k)}\Big| < \frac{\varepsilon}{3}$$

as $\mu(V) < \delta$. Similarly, the third term is $< \frac{\varepsilon}{3}$. For the second term we have

$$\Big|\int_B f_{(n_k)} - \int_B f_{(n_{k'})}\Big| = \Big|\sum_{i=1}^{m} \int_{]\alpha_i, \beta_i[} f_{(n_k)} - \sum_{i=1}^{m} \int_{]\alpha_i, \beta_i[} f_{(n_{k'})}\Big|$$

$$\leq \sum_{i=1}^{m} \Big|\int_{]\alpha_i, \beta_i[} f_{(n_k)} - \int_{]\alpha_i, \beta_i[} f_{(n_{k'})}\Big|\,.$$

As (5.4) holds for every $]\alpha_i, \beta_i[$, thus for each i the sequence $\int_{]\alpha_i, \beta_i[} f_{(n_k)}$ is convergent, so a Cauchy sequence, there exists $\overline{k}$ such that for every $k, k' \geq \overline{k}$,

$$\Big|\int_B f_{(n_k)} - \int_B f_{(n_{k'})}\Big| \leq \frac{\varepsilon}{3}$$

and, in conclusion, the sequence $\int_A f_{(n_k)}$ is a Cauchy sequence and (5.4) holds for A. If finally, A is any measurable subset of $]a, b[$, then there exists B open containing A such that $\mu(B \setminus A)$ arbitrarily small, and, proceeding like before, we can deduce (5.4) for A, from (5.4) for B. Now, we prove (5.3). Put

$$\nu(A) = \lim_{k \to \infty} \int_A f_{(n_k)}\,.$$

We will now prove that ν is a measure. We have $\nu \geq 0$ as we have assumed $f_{(n)} \geq 0$. Moreover, ν is clearly finitely additive. It remains to prove that if $A_i \subseteq [a,b]$ are measurable mutually disjoint sets, $i = 1,2,3,4,...$, then $\nu\Big(\bigcup_{i=1}^{+\infty} A_i\Big) = \sum_{i=1}^{+\infty} \nu(A_i)$. Note that by the equiintegrability of $f_{(n)}$ we easily get that

$$\forall \varepsilon > 0 \;\exists\, \delta > 0 | \Big(\mu(A) < \delta \Rightarrow \nu(A) < \varepsilon\Big) \tag{5.5}$$

For $h = 1,2,3,...$ we have $\bigcup_{i=1}^{+\infty} A_i = A_1 \cup A_2 \cup ... \cup A_h \cup \Big(\bigcup_{i=h+1}^{+\infty} A_i\Big)$, thus by the finite additivity:

$$\nu\Big(\bigcup_{i=1}^{+\infty} A_i\Big) = \Big(\sum_{i=1}^{h} \nu(A_i)\Big) + \nu\Big(\bigcup_{i=h+1}^{+\infty} A_i\Big)$$

and it suffices to prove that

$$\nu\Big(\bigcup_{i=h+1}^{+\infty} A_i\Big) \underset{h\to+\infty}{\longrightarrow} 0 \tag{5.6}$$

Note that $b - a \geq \mu\Big(\bigcup_{i=1}^{+\infty} A_i\Big) = \sum_{i=1}^{+\infty} \mu(A_i)$, then $\mu\Big(\bigcup_{i=h+1}^{+\infty} A_i\Big) = \sum_{i=h+1}^{+\infty} \mu(A_i) \underset{h\to+\infty}{\longrightarrow} 0$, and by (5.5) we deduce (5.6). Thus ν is a measure, and, clearly, it is absolutely continuous with respect to μ. By a known theorem, $\nu(A) = \int_A f$ for some $f \in L^1$. We will prove that this f satisfies (5.3). By the definition of ν, (5.3) holds when g is a characteristic function of a measurable set A. By linearity it also holds for simple measurable functions. We will now prove that (5.3) holds for any bounded measurable function g. In such a case there exist $g_{(m)}$ simple functions uniformly converging to g. Fix $\varepsilon > 0$. Then,

$$\Big|\int_{[a,b]} f_{(n_k)} g - \int_{[a,b]} fg\Big| \leq$$

$$\Big|\int_{[a,b]} f_{(n_k)} g - \int_{[a,b]} f_{(n_k)} g_{(m)}\Big| + \Big|\int_{[a,b]} f_{(n_k)} g_{(m)} - \int_{[a,b]} f g_{(m)}\Big| + \Big|\int_{[a,b]} f g_{(m)} - \int_{[a,b]} fg\Big|$$

The first term in the sum is $\leq ||g_{(m)} - g||_{L^\infty} ||f_{(n_k)}||_{L^1}$. As $f_{(n_k)}$ are bounded in L^1 and $||g_{(m)} - g||_{L^\infty} \underset{m\to+\infty}{\longrightarrow} 0$, the first term tends to 0, for $m \to +\infty$, uniformly in k, and the same consideration holds for the third term. Hence we can choose m so large that the first and the third terms are $< \frac{\varepsilon}{3}$ for every k. For such m, the second term tends to 0 for $k \to +\infty$ as $g_{(m)}$ is simple, thus, for sufficiently large k the second term is $< \frac{\varepsilon}{3}$, and (5.3) holds also for this g and the proof is complete.

I now give a different proof of the Lemma, which does not use the Radon-Nikodym Theorem, but uses the Theory of Hilbert's spaces. Again, we assume $N = 1$ and $f_{(n)} \geq 0$.

We want to work in L^2 (instead that in L^1), as L^2 is a Hilbert space. The functions $f_{(n)}$ are not necessarily in L^2, but we consider $f_{(n,m)} = \min\{f_{(n)}, m\}$ for every positive integer m. Clearly, they are in L^2 and also equiintegrable as $0 \le f_{(n,m)} \le f_{(n)}$. Moreover,

$$||f_{(n,m)}||_2 \le m\sqrt{b-a}$$

where, of course, $||f||_2$ denotes the norm of f in L^2. Hence, for every m, $f_{(n,m)}$, $n = 1, 2, 3, ...$, is a bounded sequence in L^2, and therefore it has a subsequence which is weakly convergent in L^2. By using a diagonal process, we can find a subsequence f_{n_k} of f_n such that for every positive integer m, $f_{(n_k,m)}$ weakly converges in L^2, as $k \to +\infty$, to a function $\overline{f}_{(m)}$. We have

$$f_{(n,m)} \le f_{(n,m+1)} \,.$$

Since in general, the weak limit of nonnegative functions is a.e. nonnegative,[2] from $f_{(n_k,m+1)} - f_{(n_k,m)} \ge 0$ we deduce $\overline{f}_{(m+1)} - \overline{f}_{(m)} \ge 0$ *a.e.*, hence

$$\overline{f}_{(m+1)} \ge \overline{f}_{(m)} \quad \text{a.e.}\,.$$

For the same reason, the functions $\overline{f}_{(m)}$ are ≥ 0. Moreover, they are equiintegrable, since

$$\int\limits_A \overline{f}_{(m)} = \int \overline{f}_{(m)}\chi_A = \lim_{k\to+\infty} \int f_{(n_k,m)}\chi_A = \lim_{k\to+\infty} \int\limits_A f_{(n_k,m)} \,.$$

In particular, they are bounded in L^1, so that, putting $\overline{f} = \lim_{m\to+\infty} \overline{f}_{(m)}$, we have that $\overline{f} \in L^1$ and $\int \overline{f} = \lim_{m\to+\infty} \int \overline{f}_{(m)}$ (for example by Beppo Levi's Theorem); therefore

$$||\overline{f} - \overline{f}_{(m)}||_1 \underset{m\to+\infty}{\longrightarrow} 0\,. \qquad (*)$$

We will now prove that

$$f_{(n_k)} \underset{k\to+\infty}{\longrightarrow} \overline{f} \quad \text{weakly in } L^1\,.$$

Note that putting $A_{n,m} = f_{(n)}^{-1}(]m, +\infty[)$, for some constant C we have $C \ge \int f_{(n)} \ge \int\limits_{A_{n,m}} f_{(n)} \ge m\mu(A_{n,m})$, hence $\mu(A_{n,m}) \le \frac{C}{m}$, and by the equiintegrability of $f_{(n)}$, for every $\varepsilon > 0$ there exists $\overline{m}$ such that for every $m \ge \overline{m}$ and for every n, we have

[2] Suppose $\alpha_n \ge 0$, α_n weakly converges in L^2 to α. Let $E_k := \{x : \alpha(x) \le -\frac{1}{k}\}$. Then

$$\int \alpha_n(x)\chi_{E_k} \underset{n\to+\infty}{\longrightarrow} \int \alpha(x)\chi_{E_k}$$

by the definition of weak convergence, but the left hand side integral is ≥ 0, the right hand side is $\le -\frac{1}{k}\mu(E_k)$, so that $\mu(E_k) = 0$.

$$||f_{(n)} - f_{(n,m)}||_1 = \int f_{(n)} - f_{(n,m)} = \int\limits_{A_{n,m}} f_{(n)} - m \leq \int\limits_{A_{n,m}} f_{(n)} < \varepsilon \,. \qquad (**)$$

Given $g \in L^\infty$, we have

$$\left| \int f_{(n_k)} g - \overline{f} g \right| \leq$$

$$\int |f_{(n_k)} - f_{(n_k,m)}|\, |g| + \left| \int (f_{(n_k,m)} - \overline{f}_{(m)}) g \right| + \int |\overline{f}_{(m)} - \overline{f}|\, |g| \leq$$

$$||f_{(n_k)} - f_{(n_k,m)}||_1 ||g||_\infty + \left| \int (f_{(n_k,m)} - \overline{f}_{(m)}) g \right| + ||\overline{f}_{(m)} - \overline{f}||_1 ||g||_\infty$$

Now, choose $\overline{m}$ so that $(**)$ holds for $m \geq \overline{m}$. Using (*) and (**) the first and the third summands are not greater than $\varepsilon ||g||_\infty$ for every n, provided $m \geq \overline{m}$. Fix one particular such m (for example $m = \overline{m}$), and for such an m, take k so large that the second summand is not greater than ε. This is possible, as we know that $f_{(n_k,m)}$ weakly converges to $\overline{f}_{(m)}$ in L^2, as $k \to +\infty$, using the fact that g is assumed to be in L^∞, thus also in L^2. We have concluded the proof. ∎

Lemma 5.3. *Let $f, g \in L^1([a,b], \mathbb{R}^N)$. Suppose*

$$\int_a^b f(t) \cdot v(t) + g(t) \cdot v'(t)\, dt = 0 \qquad (5.7)$$

for every $v \in Lip_0([a,b], \mathbb{R}^N)$. Then there exists $\tilde{g} \in AC([a,b], \mathbb{R}^N)$ such that $\tilde{g} = g$ a.e. on $[a,b]$ and $\tilde{g}' = f$ a.e. on $[a,b]$.

Proof. We mimic the proof of the analogous theorem in the C^1 case.
First case $f = 0$. Then (5.7) is

$$\int_a^b g(t) \cdot v'(t)\, dt = 0$$

and we also have

$$\int_a^b (g(t) - c) \cdot v'(t)\, dt = 0, \qquad (5.8)$$

for every $v \in Lip_0([a,b], \mathbb{R}^N)$, and $c \in \mathbb{R}^N$ constant, as

$$\int_a^b (g(t) - c) \cdot v'(t) = \int_a^b g(t) \cdot v'(t)\, dt - \int_a^b c \cdot v'(t)\, dt$$

and

$$\int_a^b c \cdot v'(t)\, dt = c \cdot \int_a^b v'(t)\, dt = c \cdot (v(b) - v(a)) = 0\,.$$

We could try to consider v of the form $v(t) = \int_a^t (g(s) - c)\, ds$ with c suitable constant, but such a function is AC but not necessarily Lipshitzian (note that an AC function is Lipshitzian if and only if its derivative is bounded, and the derivative of such a function v is g which is assumed to be in L^1 but not necessarily bounded). So, we modify the construction in this way. Let

$$g_{(n)}(t) = \begin{cases} g(t) & \text{if } ||g(t)|| \le n \\ 0 & \text{if } ||g(t)|| > n \end{cases}$$

$$c_n = \frac{1}{b-a} \int_a^b g_{(n)}(s) ds, \quad c_0 = \frac{1}{b-a} \int_a^b g(s) ds$$

$$v_n(t) = \int_a^t (g_{(n)}(s) - c_n) ds$$

Note that $g_{(n)} \underset{n\to+\infty}{\longrightarrow} g$ pointwise, and, as $||g_{(n)}|| \le ||g||$ (more precisely, $|(g_{(n)})_i| \le |g_i|$), we can use the dominated convergence Theorem and deduce that $c_n \underset{n\to+\infty}{\longrightarrow} c$. Moreover, for every n, v_n is Lipshitzian, and $v_n(a) = v_n(b) = 0$. Also,

$$(g_{(n)}(t) - c_n) \cdot (g(t) - c_0) =$$

$$g_{(n)}(t) \cdot g(t) + c_n \cdot c_0 - g_{(n)}(t) \cdot c_0 - g(t) \cdot c_n$$

By the definition of $g_{(n)}$, $g_{(n)} \cdot g \ge 0$; moreover, $||c_n||$ is bounded in n, as the sequence c_n is convergent, and $||g_{(n)}|| \le ||g||$, thus, using the Schwarz inequality, we have that

$$(g_{(n)}(t) - c_n) \cdot (g(t) - c_0) \ge \alpha(t) \tag{5.9}$$

for some $\alpha \in L^1$. Thus,

$$\int_a^b ||g(t) - c_0||^2\, dt = \int_a^b (g(t) - c_0) \cdot (g(t) - c_0)\, dt =$$

$$\int_a^b \lim_{n\to+\infty} (g(t) - c_0) \cdot (g_{(n)}(t) - c_n)\, dt \le \liminf_{n\to+\infty} \int_a^b (g(t) - c_0) \cdot (g_{(n)}(t) - c_n)\, dt$$

the last inequality holding as by (5.9), we can use the Fatou Lemma. Hence,

$$\int_a^b ||g(t) - c_0||^2 \, dt \le \liminf_{n\to+\infty} \int_a^b (g(t) - c_0) \cdot v_n'(t) \, dt = 0$$

by (5.8). Hence, $g(t) = c_0$ a.e.
Second Case Arbitrary f. We put

$$F(t) = \int_a^t f(s) \, ds$$

Then F is AC as f is in L^1. Moreover, by (5.7) and a partial integration, which is known to be valid for AC functions, we have

$$\int_a^b \Big(g(t) - F(t)\Big) \cdot v'(t) \, dt = 0$$

for every $v \in Lip([a,b])$ with $v(a) = v(b) = 0$, and by the first case, we have that there exists $\tilde{g} = g$ a.e. such that $\tilde{g} - F$ is constant. Therefore, $\tilde{g}'(t) = F'(t) = f(t)$ a.e.. ■

6. Calculus of variations in AC

In this Section, we study the existence and the regularity of the minimum of an integral functional. We will use the following Lemma.

Lemma 6.1. *If ϕ is a convex function of class C^1 defined on an open convex subset U in $\mathbb{R}^N$ with values in $\mathbb{R}$, then $\phi(u) \ge \phi(v) + (grad\phi)(v) \cdot (u - v)$ for every $u, v \in U$.*

Proof. Fix $u \in U$ and $v \in U$ and put $\alpha(t) = \phi(v + t(u - v))$. Then ϕ is C^1 and convex on $]-\delta, 1+\delta[$ for sufficiently small δ. Hence

$$\alpha(1) \ge \alpha(0) + (1-0)\alpha'(0) \, .$$

It now suffices to note that $\alpha(0) = \phi(v)$, $\alpha(1) = \phi(u)$, $\alpha'(0) = (grad\phi)(v) \cdot (v - u)$. ■

Let

$$J(y) = \int_a^b L\big(t, y(t), y'(t)\big) \, dt \, .$$

where $L : [a,b] \times \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is assumed to be continuous and Ω is a nonempty open connected subset of $\mathbb{R}^N$.

Theorem 6.2. *Suppose J and L are as above. Suppose also*

i) $L = L(t, y, q)$ is convex with respect to q
ii) $L(t, y, q) \geq \theta(||q||) - c$ for all $t \in [a, b], y \in \overline{\Omega}, q \in \mathbb{R}^N$, where $c \geq 0$ and $\theta : [0, +\infty[\to \mathbb{R}$ satisfies $\frac{\theta(r)}{r} \underset{r \to +\infty}{\longrightarrow} +\infty$, $\theta \geq 0$.
iii) $\frac{\partial L}{\partial q}$ exists and is continuous on $[a, b] \times \overline{\Omega} \times \mathbb{R}^N$.
Then the problem

$$\min\Big\{J(y) : y \in F_{A,B}\Big\}, \quad F_{A,B} := \Big\{y \in AC([a,b], \overline{\Omega}) : y(a) = A, y(b) = B\Big\} \qquad P_{A,B}$$

has a solution for every $A, B \in \Omega$. Here, of course $y \in AC([a, b], \overline{\Omega})$ means that $y \in AC([a, b], \mathbb{R}^N)$, and $y(t) \in \overline{\Omega}$ for every $t \in [a, b]$.

Proof. Note that for every $y \in F_{A,B}$ we have $J(y) \geq -c(b-a)$, so J is bounded from below. On the other hand, we have $J(y) < +\infty$ if y is C^1 as in such case the set $\{(t, y(t), y'(t)) : t \in [a, b]\}$ lies in a compact where L is bounded, but we could have $J(y) = +\infty$ for some $y \in F_{A,B}$, as in such a case y' is in L^1, and if for example $L(t, y, q) = q^2$, then $J(y)$ is finite only when y' is in L^2. In any case, as Ω is assumed to be connected, there exists y C^1 which lies in $F_{A,B}$, so that $\inf\Big\{J(y) : y \in F_{A,B}\Big\} < +\infty$.

First step. Let $y_{(n)} \in F_{A,B}$ be so that $J(y_{(n)}) \underset{n \to \infty}{\longrightarrow} \inf\Big\{J(y) : y \in F_{A,B}\Big\}$. We prove that $y'_{(n)}$ are equiintegrable.

For every $M > 0$ let be given $K_M > 0$ so that $\frac{\theta(r)}{r} \geq M$ if $r \geq K_M$. Let E be a measurable subset of $[a, b]$, let

$$E_1 = \{t \in E : ||y'_{(n)}(t)|| \geq K_M\}, E_2 = \{t \in E : ||y'_{(n)}(t)|| < K_M\}$$

Then

$$\begin{aligned}\int_E ||y'_{(n)}(t)||\,dt &= \int_{E_1} ||y'_{(n)}(t)||\,dt + \int_{E_2} ||y'_{(n)}(t)||\,dt \\ &\leq \int_{E_1} \frac{\theta\big(||y'_{(n)}(t)||\big)}{M}\,dt + K_M \mu(E_2) \\ &\leq \frac{1}{M}\int_a^b \theta\Big(||y'_{(n)}(t)||\Big)\,dt + K_M\mu(E) \\ &\leq \frac{1}{M}\Big(\int_a^b L\big(t, y_{(n)}(t), y'_{(n)}(t)\big) + c\,dt\Big) + K_M\mu(E) \\ &= \frac{1}{M}\big(J(y_{(n)}) + c(b-a)\big) + K_M\mu(E)\end{aligned}$$

and, as the sequence $J(y_{(n)})$ is convergent, thus bounded, for every $\varepsilon > 0$ for sufficiently large M independent of n, we have $\frac{1}{M}\big(J(y_{(n)}) + c(b-a)\big) < \frac{\varepsilon}{2}$, and for such M we find $\delta > 0$ such that if $\mu(E) < \delta$, then $K_M\mu(E) < \frac{\varepsilon}{2}$, so that the first step is proved.

Second Step. There exists a subsequence $y_{(n_k)}$ *of* $y_{(n)}$ *and* $\overline{y}$ *AC such that* $y_{(n_k)} \underset{k\to+\infty}{\longrightarrow} \overline{y}$ *uniformly and* $y'_{(n_k)} \underset{k\to+\infty}{\longrightarrow} \overline{y}'$ *weakly in* L^1.

By Lemmas 5.1 and 5.2 we find a subsequence subsequence $y_{(n_k)}$ of $y_{(n)}$, $\overline{y}$ AC, $\eta \in L^1$ such that $y_{(n_k)} \underset{k\to+\infty}{\longrightarrow} \overline{y}$ uniformly and $y'_{(n_k)} \underset{k\to+\infty}{\longrightarrow} \eta$ weakly in L^1. To complete the proof of the second step it remains to prove that $\overline{y}' = \eta$ a.e. Let $v \in Lip_0([a,b],\mathbb{R}^N)$. Then, recalling that the scalar product of AC functions is an AC function, we have

$$\int_a^b v \cdot y'_{(n_k)} + v' \cdot y_{(n_k)} = \int_a^b (v \cdot y_{(n_k)})' = (v \cdot y_{(n_k)})(b) - (v \cdot y_{(n_k)})(a) = 0$$

By taking the limit, we get

$$\int_a^b v \cdot \eta + v' \cdot \overline{y} = 0$$

thus, by Lemma 5.3 we have $\tilde{y}' = \eta$ a.e. where $\tilde{y}$ is AC and $\overline{y} = \tilde{y}$ a.e., but, as the complement of a set of 0 measure is dense and $\overline{y}$ and $\tilde{y}$ are both continuous, then $\overline{y} = \tilde{y}$. This concludes the proof of the second step.

Third step. $\overline{y}$ *is a solution of* $P_{A,B}$. It clearly suffices to prove that

$$J(\overline{y}) \leq \liminf_{k\to+\infty} J(y_{(n_k)}) \tag{6.1}$$

By Lemma 6.1 we have

$$L(t, y_{(n_k)}(t), y'_{(n_k)}(t)) \geq L(t, y_{(n_k)}(t), \overline{y}'(t)) + \frac{\partial L}{\partial q}(t, y_{(n_k)}(t), \overline{y}'(t)) \cdot \big(y'_{(n_k)}(t) - \overline{y}'(t)\big) \tag{6.2}$$

The idea is now to integrate over $[a,b]$ and to take the liminf on both the sides. However, in order to keep $\overline{y}'$ bounded it is more convenient to integrate over the sets A_r defined by

$$A_r = \big\{t \in [a,b] : ||\overline{y}'(t)|| \leq r\big\}$$

By (6.2), we have

$$\int_{A_r} L(t, y_{(n_k)}(t), y'_{(n_k)}(t))\, dt \geq$$

$$\int_{A_r} L(t, y_{(n_k)}(t), \overline{y}'(t))\, dt + \int_{A_r} \frac{\partial L}{\partial q}(t, \overline{y}(t), \overline{y}'(t)) \cdot \big(y'_{(n_k)}(t) - \overline{y}'(t)\big)\, dt +$$

$$\int_{A_r} \Big(\frac{\partial L}{\partial q}(t, y_{(n_k)}(t), \overline{y}'(t)) - \frac{\partial L}{\partial q}(t, \overline{y}(t), \overline{y}'(t))\Big) \cdot \big(y'_{(n_k)}(t) - \overline{y}'(t)\big)\, dt\,.$$

We have

$$\int\limits_{A_r} \frac{\partial L}{\partial q}(t, \overline{y}(t), \overline{y}'(t)) \cdot \big(y'_{(n_k)}(t) - \overline{y}'(t)\big)\, dt =$$

$$\int\limits_{[a,b]} \chi_{A_r}(t) \frac{\partial L}{\partial q}(t, \overline{y}(t), \overline{y}'(t)) \cdot \big(y'_{(n_k)}(t) - \overline{y}'(t)\big)\, dt \underset{k\to+\infty}{\longrightarrow} 0$$

as the function $\chi_{A_r}(t)\frac{\partial L}{\partial q}(t, \overline{y}(t), \overline{y}'(t))$ is bounded, since $\frac{\partial L}{\partial q}$ is continuous by hypothesis so is bounded on the compact set $[a,b] \times \Big\{\overline{y}(t) : t \in [a,b]\Big\} \times \overline{B(0,r)}$, and in view of the second step. Note that $\overline{y}$ is continuous, so bounded on $[a,b]$, but $\overline{y}'$ is not necessarily continuous, so we have to restrict our considerations to A_r. Moreover,

$$\int\limits_{A_r} \Big(\frac{\partial L}{\partial q}(t, y_{(n_k)}(t), \overline{y}'(t)) - \frac{\partial L}{\partial q}(t, \overline{y}(t), \overline{y}'(t))\Big) \cdot \big(y'_{(n_k)}(t) - \overline{y}'(t)\big)\, dt \underset{k\to+\infty}{\longrightarrow} 0 \tag{6.3}$$

To prove (6.3), note that

$$||y'_{(n_k)}||_{L^1} \leq C_1 \tag{6.4}$$

for some constant C_1 as $y'_{(n_k)}$ are equiintegrable; another way of seeing that is that $y'_{(n_k)}$ is a weakly convergent sequence and every weakly convergent sequence is bounded. Moreover, for some constant C_2 we have

$$||\overline{y}(t)|| \leq C_2$$

as the set $\{\overline{y}(t) : t \in [a,b]\}$ is compact. Hence,

$$||y_{(n_k)}(t)|| \leq ||\overline{y}(t)|| + ||y_{(n_k)}(t) - \overline{y}(t)|| \leq C_2 + 1$$

for sufficiently large k as $y_{(n_k)}$ tends to $\overline{y}$ uniformly. In conclusion, for $t \in A_r$ we have

$$(t, y_{(n_k)}(t), \overline{y}'(t)), (t, \overline{y}(t), \overline{y}'(t)) \in [a,b] \times \Big(\overline{B(0, C_2+1)} \cap \overline{\Omega}\Big) \times \overline{B(0,r)}$$

which is a compact set where $\frac{\partial L}{\partial q}$ is uniformly continuous. As for every $\delta > 0$ we have

$$||(t, y_{(n_k)}(t), \overline{y}'(t)) - (t, \overline{y}(t), \overline{y}'(t))|| < \delta$$

for sufficiently large k, we conclude that

$$\frac{\partial L}{\partial q}(t, y_{(n_k)}(t), \overline{y}'(t)) \underset{k\to+\infty}{\longrightarrow} \frac{\partial L}{\partial q}(t, \overline{y}(t), \overline{y}'(t))$$

uniformly, and in view of (6.4), (6.3) easily follows. As L is continuous, by a similar reason,

$$\int\limits_{A_r} L(t, y_{(n_k)}(t), \overline{y}'(t))\, dt \underset{k\to+\infty}{\longrightarrow} \int\limits_{A_r} L(t, \overline{y}(t), \overline{y}'(t))\, dt$$

In conclusion, if we would integrate over $[a,b]$ instead of over A_r, (6.1) would be proved. We conclude observing that

$$\liminf_{k\to+\infty} J(y_{(n_k)}) = \liminf_{k\to+\infty} \int_{[a,b]} (L+c)(t, y_{(n_k)}(t), y'_{(n_k)}(t))\, dt - \int_{[a,b]} c\, dt$$

$$\geq \liminf_{k\to+\infty} \int_{A_r} (L+c)(t, y_{(n_k)}(t), y'_{(n_k)}(t))\, dt - \int_{[a,b]} c\, dt$$

$$\geq \int_{A_r} (L+c)(t, \overline{y}(t), \overline{y}'(t))\, dt - \int_{[a,b]} c\, dt$$

for every $r = 1, 2, 3, ...$ As the sequence of sets A_r is increasing and the union of such sets is $[a,b]$ minus a set of 0 measure, passing to the limit, we have

$$\liminf_{k\to+\infty} J(y_{(n_k)}) \geq \int_{[a,b]} (L+c)(t, \overline{y}(t), \overline{y}'(t))\, dt - \int_{[a,b]} c\, dt = J(\overline{y}),$$

and the Theorem is proved. Note that we have worked on the integral of $L+c$ instead of that of L, in order to have a nonnegantive integrand. ∎

It is possible in fact to prove that the hypothesis iii) can be removed in Theorem 6.2. I remark that the previous theorem shows the use of the weak convergence. We now want to study when the minimum that is stated to exists is in fact of class C^1, in other words, we aim to get a regularity result.

Theorem 6.3. *Suppose the hypotheses of Theorem 6.3 are satisfied. Suppose moreover, that* $\frac{\partial L}{\partial y}$ *exists and is continuous on* $[a,b] \times \Omega \times \mathbb{R}^N$ *and also,*
i) There exist positive constants $C_1(r)$, $C_2(r)$ *such that*

$$\max\left\{ \left|\left| \frac{\partial L}{\partial y}(t,y,q) \right|\right|, \left|\left| \frac{\partial L}{\partial q}(t,y,q) \right|\right| \right\} \leq C_1(r)\theta(||q||) + C_2(r)$$

for every $t \in [a,b]$ *and* $y \in \Omega \cap \overline{B(0,r)}$.
ii) For every $M \geq 0$ *there exists* $K_M > 0$ *such that*

$$\theta(r') \leq K_M(1 + \theta(r)) \text{ if } r, r' \geq 0, r' \leq r + M.$$

Then every AC solution $\overline{y}$ *of* $P_{A,B}$ *with* $\overline{y}(t) \in \Omega$, *for all* $t \in [a,b]$ *satisfies the Euler equation in the sense that the function* α *defined by* $\alpha(t) = \frac{\partial L}{\partial q}(t, \overline{y}(t), \overline{y}'(t))$ *amounts a.e. to an AC function* $\tilde{\alpha}$ *and* $\tilde{\alpha}'(t) = \frac{\partial L}{\partial y}(t, \overline{y}(t), \overline{y}'(t))$ *a.e. Moreover,* $\overline{y}$ *is Lipshitzian.*

Proof. Assume for the moment that $\overline{y}$ is an AC function with values in Ω with $J(\overline{y}) < +\infty$ (for this consideration we do not need that $\overline{y}$ is a minimum point for J). Let $v \in Lip_0([a,b], \mathbb{R}^N)$. We put

$$G(\lambda, t) = L\big(t, \overline{y}(t) + \lambda v(t), \overline{y}'(t) + \lambda v'(t)\big) \quad \lambda \in [-\delta, \delta]$$

where $\delta > 0$ is chosen so that, the set $F := \{\overline{y}(t) + \lambda v(t) : t \in [a,b], \lambda \in [-\delta,\delta]\}$, is contained in Ω. Note that F is compact, as the continuous image of the set $[a,b] \times [-\delta,\delta]$ via a continuous function. We now prove that there exists a summable function u such that for every $t \in [a,b]$ and $\lambda \in [-\delta,\delta]$ we have

$$\Big|\Big|\frac{\partial L}{\partial y}(t,\overline{y}(t) + \lambda v(t), \overline{y}'(t) + \lambda v'(t))\Big|\Big| \leq u(t) \tag{6.5}$$

$$\Big|\Big|\frac{\partial L}{\partial q}(t,\overline{y}(t) + \lambda v(t), \overline{y}'(t) + \lambda v'(t))\Big|\Big| \leq u(t) \tag{6.6}$$

We have

$$\Big|\Big|\frac{\partial L}{\partial y}(t,\overline{y}(t) + \lambda v(t), \overline{y}'(t) + \lambda v'(t))\Big|\Big| \leq C_1(r)\theta\big(||\overline{y}'(t) + \lambda v'(t)||\big) + C_2(r)$$

$$\leq C_1(r)\Big(K_M(1 + \theta(||\overline{y}'(t)||))\Big) + C_2(r)$$

where r is such that $F \subseteq \overline{B(0,r)}$ and $M = \delta \sup\{||v'(t)|| : t \in [a,b]\}$. Now, $\theta(||\overline{y}'(t)||) \leq L(t,\overline{y}(t),\overline{y}'(t)) + c$, and $L(t,\overline{y}(t),\overline{y}'(t))$ has finite integral $= J(\overline{y})$, so that (6.5) easily follows, and (6.6) can be proved in the same way. We also deduce

$$\Big|\frac{\partial G}{\partial \lambda}(\lambda,t)\Big| \leq \tilde{u}(t) := u(t)\big(||v(t)|| + ||v'(t)||\big) \tag{6.7}$$

and, as v and v' are bounded, then $\tilde{v}$ is summable. In particular, as

$$G(\lambda,t) = G(0,t) + \lambda\frac{\partial G}{\partial \lambda}(\mu_{\lambda,t},t)$$

where $\mu_{\lambda,t} \in]0,\lambda[$, and $G(0,t) = L\big(t,\overline{y}(t),\overline{y}'(t)\big)$ is summable with respect to $t \in [a,b]$, then $J(\overline{y} + \lambda v) < +\infty$. Moreover, by (6.7), putting $g(\lambda) = \int_{[a,b]} G(\lambda,t)\, dt$, we can differentiate g, by differentiating G in the integral. Thus, supposing now that $\overline{y}$ is an AC solution of $P_{A,B}$, we have

$$0 = g'(0) = \int_a^b \frac{\partial L}{\partial y}(t,\overline{y}(t),\overline{y}'(t)) \cdot v(t) + \frac{\partial L}{\partial q}(t,\overline{y}(t),\overline{y}'(t)) \cdot v'(t)$$

and the Euler equation follows from Lemma 5.3. To prove that $\overline{y}$ is Lipshitzian, note that, by i) and ii) in Theorem 6.2 we have

$$\theta(||\overline{y}'(t)||) - c \leq L(t,\overline{y}(t),\overline{y}'(t)) \leq L(t,\overline{y}(t),0) + \frac{\partial L}{\partial q}(t,\overline{y}(t),\overline{y}'(t)) \cdot \overline{y}'(t)$$

so that we have a.e.

$$\theta(||\overline{y}'(t)||) \leq c + L(t,\overline{y}(t),0) + ||\tilde{\alpha}(t)||\,||\overline{y}'(t)|| \leq K_1||\overline{y}'(t)|| + K_2$$

as $\{(t, \overline{y}(t), 0) : t \in [a, b]\}$ is compact and $\tilde{\alpha}$ is AC, we have a.e. either $\overline{y}'(t) = 0$ or

$$\frac{\theta(||\overline{y}'(t)||)}{||\overline{y}'(t)||} \leq K_1 + \frac{K_2}{||\overline{y}'(t)||}$$

If $||\overline{y}'(t)||$ is sufficiently large, say if $||\overline{y}'(t)|| > K_3$, then the right hand side is $< K_1 + 1$ and the left hand side is $> K_1 + 1$, so that we must have $||\overline{y}'(t)|| \leq K_3$ a.e., and thus, using the fundamental theorem of integral calculus for AC functions, $\overline{y}$ is Lipshitzian. ■

An example of a function θ that satisfies the hypotheses of Theorems 6.2 and 6.3 is given by $\theta(r) = r^k$, $k > 1$. I prove that it satisfies ii) in Theorem 6.3. We in fact have

$$r'^k \leq (M+1)^k (r^k + 1) \text{ if } r, r' \geq 0, r' \leq r + M\,, \tag{6.8}$$

as we can assume $r' = r + M$, and we have thus to prove $(r + M)^k \leq (r^k + 1)(M + 1)^k$, and this is trivial if $r \leq 1$, and follows from the simple inequality $r + M \leq r(1 + M)$ if $r > 1$. The same considerations are valid for $\theta(r) = \alpha r^k$, $k > 1$, $\alpha > 0$. In general, in fact, we can prove that if θ satisfies the hypotheses of Theorems 6.2 and 6.3, then so does $\alpha\theta$ with $\alpha > 0$. To see this, note that

$$\alpha\theta(r') \leq \alpha K_M (1 + \theta(r)) \leq (1 + \alpha) K_M (1 + \alpha\theta(r))$$

if $r, r' \geq 0, r' \leq r + M$.

Theorem 6.4. *In the hypothesis of Theorem 6.3, if moreover the map* $q \mapsto \frac{\partial L}{\partial q}(t, \overline{y}(t), q)$ *is one-to-one for every* $t \in [a, b]$ *(this is the case if* L *is strictly convex with respect to* q*), then* $\overline{y}$ *is of class* C^1.

Proof. Let

$$E := \left\{ t \in [a, b] : \nexists\, \overline{y}'(t) \text{ or } \tilde{\alpha}(t) \neq \alpha(t) \right\}$$

We know that E has measure 0, so that $[a, b] \setminus E$ is dense in $[a, b]$, hence every point in $[a, b]$ is the limit of a sequence of points in $[a, b] \setminus E$. Let $\overline{t} \in [a, b]$ and suppose $t_k \underset{k \to +\infty}{\longrightarrow} \overline{t}$, $\overline{y}'(t_k) \underset{k \to +\infty}{\longrightarrow} x$, $t_k \in [a, b] \setminus E$. As $\frac{\partial L}{\partial q}$ is assumed to be continuous, we have

$$\frac{\partial L}{\partial q}(\overline{t}, \overline{y}(\overline{t}), x) = \lim_{k \to +\infty} \frac{\partial L}{\partial q}(t_k, \overline{y}(t_k), \overline{y}'(t_k)) = \lim_{k \to +\infty} \tilde{\alpha}(t_k) = \tilde{\alpha}(\overline{t})$$

By the hypothesis of injectivity of $q \mapsto \frac{\partial L}{\partial q}(t, \overline{y}(t), q)$, we see that x does not depend of the sequence t_k, in other words that there exists $\overline{x} \in \mathbb{R}$ such that, if $t_k \underset{k \to +\infty}{\longrightarrow} \overline{t}$, $\overline{y}'(t_k) \underset{k \to +\infty}{\longrightarrow} x$, $t_k \in [a, b] \setminus E$, then $x = \overline{x}$. We could still suspect, however, that for some $t_k \underset{k \to +\infty}{\longrightarrow} \overline{t}$, $t_k \in [a, b] \setminus E$, the sequence $\overline{y}'(t_k)$ is not convergent, but this is not the case. In fact, as $\overline{y}$ is Lipshitzian, the sequence $\overline{y}'(t_k)$ is bounded, so it has a convergent subsequence, and its limit amounts to $\overline{x}$, hence every subsequence of $\overline{y}'(t_k)$ has a subsequence convergent to $\overline{x}$, and by a simple and well-known result in general topology, this implies that the sequence $\overline{y}'(t_k)$ is convergent to $\overline{x}$. We now prove that $\overline{x} = \overline{y}'(\overline{t})$, i.e. $\overline{x}_i = \overline{y}_i'(\overline{t})$ for every $i = 1, ..., N$. We have concluded that for every $t_k \underset{k\to+\infty}{\longrightarrow} \overline{t}$ with $t_k \in [a,b] \setminus E$, we have $\overline{y}'(t_k) \underset{k\to+\infty}{\longrightarrow} \overline{x}$, hence

$$\overline{y}'(t)|_{[a,b]\setminus E} \underset{t\to\overline{t}}{\longrightarrow} \overline{x} \tag{6.9}$$

We have

$$\frac{\overline{y}_i(t) - \overline{y}_i(\overline{t})}{t - \overline{t}} = \frac{\int_{\overline{t}}^{t} \overline{y}_i'(s)\, ds}{t - \overline{t}} \tag{6.10}$$

and, if $t > \overline{t}$ we have

$$(t-\overline{t}) \inf\{\overline{y}_i'(s) : s \in]\overline{t}, t[\setminus E\} \leq \int_{\overline{t}}^{t} \overline{y}_i'(s)\, ds \leq (t-\overline{t}) \sup\{\overline{y}_i'(s) : s \in]\overline{t}, t[\setminus E\}$$

as $\mu(E) = 0$, hence by (6.9), for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $t \in [a,b] \cap]\overline{t}, \overline{t}+\delta[$ we have

$$\overline{x} - \varepsilon < \inf\{\overline{y}_i'(s) : s \in]\overline{t}, t[\setminus E\} \leq \sup\{\overline{y}_i'(s) : s \in]\overline{t}, t[\setminus E\} < \overline{x} + \varepsilon$$

and by (6.10), if $\overline{t} < b$, we have

$$\frac{\overline{y}_i(t) - \overline{y}_i(\overline{t})}{t - \overline{t}} \underset{t\to\overline{t}^+}{\longrightarrow} \overline{x}_i$$

We conclude similarly that, if $\overline{t} > a$,

$$\frac{\overline{y}_i(t) - \overline{y}_i(\overline{t})}{t - \overline{t}} \underset{t\to\overline{t}^-}{\longrightarrow} \overline{x}_i$$

noting that

$$\frac{\overline{y}_i(t) - \overline{y}_i(\overline{t})}{t - \overline{t}} = \frac{\int_{t}^{\overline{t}} \overline{y}_i'(s)\, ds}{\overline{t} - t}$$

and proceeding like before, but assuming $t < \overline{t}$. In conclusion, $\overline{y}'(\overline{t}) = \overline{x}$, as claimed, or in other words, for every $t \in [a,b]$, $\overline{y}'(t) = \lim_{k\to+\infty} \overline{y}'(t_k)$, where t_k is any sequence tending to t with $t_k \in [a,b] \setminus E$. It remains to prove that $\overline{y}'$ is continuous. Note that if E would be the set of points in $[a,b]$ at which the derivative does not exist, then the proof of the continuity of $\overline{y}'$ would be trivial as at this point it would follow $E = \emptyset$. As however the definition of E is a bit more complicated we need another proof. Let $t_n \underset{n\to+\infty}{\longrightarrow} t$, $t_n \in [a,b]$. We have to prove

$$\overline{y}'(t_n) \underset{n\to+\infty}{\longrightarrow} \overline{y}'(t). \tag{6.11}$$

By what we have proved, for every n there exists a sequence $t_{n,k} \in [a,b] \setminus E$ such that $t_{n,k} \underset{k\to+\infty}{\longrightarrow} t_n$ and $\overline{y}'(t_{n,k}) \underset{k\to+\infty}{\longrightarrow} \overline{y}'(t_n)$. Therefore, for every n there exists k_n such that

$$|t_{n,k_n} - t_n| < \frac{1}{n} \tag{6.12}$$

$$||\overline{y}'(t_{n,k_n}) - \overline{y}'(t_n)|| < \frac{1}{n} \tag{6.13}$$

By (6.12) we have $t_{n,k_n} \underset{n\to+\infty}{\longrightarrow} t$. Hence, we know that $\overline{y}'(t_{n,k_n}) \underset{n\to+\infty}{\longrightarrow} \overline{y}'(t)$. Hence by (6.13), (6.11) holds. ■

An example in which all hypotheses of Theorems 6.2, 6.3, 6.4 hold is when $N = 1$, $\Omega = \mathbb{R}$, and

$$L(t,y,q) = \alpha(t,y)|q|^k + \beta(t,y) \tag{6.14}$$

with $k > 1$, α and β of class C^1 on $[a,b] \times \mathbb{R}$ (it suffices in fact that α and β are continuous with their derivatives with respect to y), and $\alpha(t,y) \geq \overline{\alpha}$, $\beta(t,y) \geq \overline{\beta}$ for every $(t,y) \in [a,b] \times \mathbb{R}$, and $\overline{\alpha} > 0$. We can choose in this case $\theta(r) = \overline{\alpha} r^k$. Note that

$$\frac{\partial L}{\partial q}(t,y,q) = \alpha(t,y)k|q|^{k-1}\mathrm{sign}(q)$$

$$\frac{\partial L}{\partial y}(t,y,q) = \frac{\partial \alpha}{\partial y}(t,y)|q|^k + \frac{\partial \beta}{\partial y}(t,y)$$

Observing that the functions $\frac{\partial \alpha}{\partial y}$ and $\frac{\partial \beta}{\partial y}$ are continuous by hypothesis, thus bounded on every bounded subset of $[a,b] \times \mathbb{R}$, and that $r^{k-1} \leq r^k + 1$, we see that i) of Theorem 6.3 holds. Moreover, the function $q \mapsto |q|^{k-1}\mathrm{sign}(q)$ is strictly increasing as can be easily verified, hence so is the map $q \mapsto \alpha(t,y)k|q|^{k-1}\mathrm{sign}(q)$ for every $(t,y) \in [a,b] \times \mathbb{R}$, hence also the hypothesis of Theorem 6.4 is satisfied. It follows that for L as in (6.14) J has a minimum in $F_{A,B}$ for every $A, B \in \mathbb{R}$ and that every such a minimum is of class C^1.

7. Geodesics on Manifolds Again.

In this Section we return on geodesics and use the results of Section 6. First of all, given a continuous curve $\gamma : [a,b] \to \mathbb{R}^N$ with $L(\gamma) := L < +\infty$ ($L(\gamma)$ denoting the length of γ), we want to construct an "equivalent" curve $\tilde{\gamma}$ parametrized by arclength. I prefer not to specify what is the exact meaning of the word *equivalent*. It would be nice if $\tilde{\gamma}$ was a reparametrization of γ, but unfortunately this is not possible in general, as, if γ is constant in some interval, then so is any reparametrization of it. Let $\Phi : [a,b] \to [0,L]$ be defined by $\Phi(t) = L(\gamma|_{[0,t]})$. Then, as seen in Section 3, Φ is increasing and continuous, hence its image is the interval $[0, L]$. We define $\tilde{\gamma} : [0, L] \to \mathbb{R}^N$ by

$$\tilde{\gamma}(\tau) = \gamma(t) : \Phi(t) = \tau. \tag{7.1}$$

However, as Φ is not, a priori, one-to-one, we have to prove that such a definition is correct, i.e., that it does not depend on $t \in \Phi^{-1}(\{\tau\})$, in other words, that if $\Phi(t_1) = \Phi(t_2) = \tau$, then $\gamma(t_1) = \gamma(t_2)$. Suppose for example $t_1 \leq t_2$. Then

$$L(\gamma|_{[t_1,t_2]}) = L(\gamma|_{[a,t_2]}) - L(\gamma|_{[a,t_1]}) = \Phi(t_2) - \Phi(t_1) = 0\,. \tag{7.2}$$

Therefore, γ is constant in $[t_1, t_2]$ and $\gamma(t_1) = \gamma(t_2)$, as claimed. Note that by definition we have $\gamma = \tilde{\gamma} \circ \Phi$, hence Φ being onto $[0, L]$, γ and $\tilde{\gamma}$ have the same image. Next, we prove that $\tilde{\gamma}$ is continuous. Let $t_1, t_2 \in [a, b]$, $t_1 \leq t_2$. Then, $|\Phi(t_2) - \Phi(t_1)| = L(\gamma|_{[t_1,t_2]}) \geq ||\gamma(t_2) - \gamma(t_1)||$. Hence, if $\tau_1, \tau_2 \in [0, L]$ and for example $\tau_1 < \tau_2$, taking t_1, t_2 so that $\Phi(t_1) = \tau_1$ and $\Phi(t_2) = \tau_2$, we have $t_1 < t_2$ as if $t_1 \geq t_2$, we would have $\tau_1 = \Phi(t_1) \geq \Phi(t_2) = \tau_2$. Also

$$||\tilde{\gamma}(\tau_2) - \tilde{\gamma}(\tau_1)|| = ||\gamma(t_2) - \gamma(t_1)|| \leq |\Phi(t_2) - \Phi(t_1)| = |\tau_2 - \tau_1| \tag{7.3}$$

so that $\tilde{\gamma}$ is 1-Lipshitzian, hence continuous. We are now going to prove that, if $t_1, t_2 \in [a, b]$, $t_1 < t_2$, then

$$L\Big(\tilde{\gamma}|_{[\Phi(t_1),\Phi(t_2)]}\Big) = L\Big(\gamma|_{[t_1,t_2]}\Big) \tag{7.4}$$

so that, also using (7.2), we have $L\Big(\tilde{\gamma}|_{[\Phi(t_1),\Phi(t_2)]}\Big) = \Phi(t_2) - \Phi(t_1)$, i.e., $\tilde{\gamma}$ is parametrized by arclength. Note that, taking $t_1 = a$, $t_2 = b$, we see from (7.4) that $\tilde{\gamma}$ and γ have the same length. We now prove (7.4). If $\Phi(t_1) = \Phi(t_2)$ (7.4) is trivial in view of (7.2), thus we can and do assume $\Phi(t_1) < \Phi(t_2)$. Let $(a_0, a_1, ..., a_n) \in \mathcal{P}_{t_1,t_2}$. Then $\Lambda_\gamma(a_0, a_1, ..., a_n) = \Lambda_{\tilde{\gamma}}(\Phi(a_0), \Phi(a_1), ..., \Phi(a_n))$. Note that, strictly speaking, $(\Phi(a_0), \Phi(a_1), ..., \Phi(a_n))$ is not a necessarily a partition of $[\Phi(t_1), \Phi(t_2)]$, as Φ being increasing but not strictly increasing, we could have $\Phi(t_{i-1}) = \Phi(t_i)$. However, we can easily replace it with a partition Π of $[\Phi(t_1), \Phi(t_2)]$ in the following way. The first element is $\Phi(t_1) = b_0$, the second is the first element b_1 of the form $\Phi(t_i)$ bigger than $\Phi(t_1)$, the third is the first element of the form $\Phi(t_i)$ bigger than b_1 and so on. Then, we have

$$\Lambda_\gamma(a_0, a_1, ..., a_n) = \Lambda_{\tilde{\gamma}}(\Phi(a_0), \Phi(a_1), ..., \Phi(a_n)) = \Lambda_{\tilde{\gamma}}(\Pi)\,.$$

Conversely, if $(b_0, b_1, ..., b_n) \in \mathcal{P}_{\Phi(t_1),\Phi(t_2)}$, clearly $b_0 = \Phi(t_1)$, $b_n = \Phi(t_2)$; let $a_i \in [a, b]$ be so that $\Phi(a_i) = b_i$, and choose $a_0 = t_1$, $a_n = t_2$. Then $a_{i-1} < a_i$ so that $(a_0, a_1, ..., a_n) \in \mathcal{P}_{t_1,t_2}$. Moreover, $\Lambda_\gamma(a_0, a_1, ..., a_n) = \Lambda_{\tilde{\gamma}}(b_0, b_1, ..., b_n)$. Now, (7.4) is an immediate consequence of the definition of the length of a curve. We summarize the results obtained in the following Theorem.

Theorem 7.1. *The curve $\tilde{\gamma}$ is correctly defined by (7.1), is continuous, parametrized by arclength, and have the same length and the same image as γ.* ■

Lemma 7.2. (Jensen's inequality). *Let ϕ be a convex function of class C^1 from $\mathbb{R}^N$ to $\mathbb{R}$. Let g be a summable function from the interval $[a,b]$ with values in $\mathbb{R}^N$. Then we have:*

$$\frac{1}{b-a}\int_a^b \phi(g) \geq \phi\Big(\frac{1}{b-a}\int_a^b g\Big).$$

Proof. We use Lemma 6.1 with $v = \frac{1}{b-a}\int_a^b g$, and put $L(u) = \phi(v) + (grad\phi)(v)\cdot(u-v)$. We have $\phi(u) \geq L(u)$ and the inequality holds for $u = v$. Then,

$$\phi\Big(\frac{1}{b-a}\int_a^b g\Big) = \phi(v) = L(v) =$$

$$L\Big(\frac{1}{b-a}\int_a^b g\Big) = \frac{1}{b-a}\int_a^b L(g) \leq \frac{1}{b-a}\int_a^b \phi(g)$$

where we have used the following simple remark: As L has the form $L(u) = c\cdot u + d$, then

$$L\Big(\frac{1}{b-a}\int_a^b g\Big) = c\cdot\Big(\frac{1}{b-a}\int_a^b g\Big) + d = \Big(\frac{1}{b-a}\int_a^b c\cdot g\Big) + \frac{1}{b-a}\int_a^b d = \frac{1}{b-a}\int_a^b L(g).$$

■

Remark 7.3. The hypothesis that ϕ is of class C^1 is in fact not necessary. This, as it can be proved that for any convex function ϕ from $\mathbb{R}^N$ to $\mathbb{R}$ and for every $v \in \mathbb{R}^N$ there exists L of the form $L(u) = c\cdot u + d$ such that $L(u) \leq \phi(u)$ and the equality holds for $u = v$. Such a statement is a consequence of the Hahn-Banach Theorem. In the particular case $N = 1$, it is simple to verify that we can take $L(u) = \phi(v) + \phi'_+(v)(u-v)$ where $\phi'_+(v)$ denotes the right derivative of ϕ at v. Recall that a convex function from an open subset of $\mathbb{R}$ with values in $\mathbb{R}$ has both right and left derivative at all points. ■

Remark 7.4. The Jensen inequality is useful in many cases. We will use it for $N = 1$, $\phi(x) = x^2$. Note that in this particular case, the inequality is also a consequence of the Holder inequality:

$$\Big|\int_a^b g\Big| \le \int_a^b |g| \le \Big(\int_a^b 1\Big)^{\frac{1}{2}}\Big(\int_a^b g^2\Big)^{\frac{1}{2}} \Longrightarrow \Big(\int_a^b g\Big)^2 \le (b-a)\int_a^b g^2 . \quad \blacksquare$$

We need to know that the usual formula for the length of C^1 (or more generally piecewise C^1) curves is still valid for AC curves.

Lemma 7.5. *Let γ be an AC curve from $[a,b]$ to $\mathbb{R}^N$. Then $L(\gamma) = \int_a^b ||\gamma'(t)||\, dt$.*

Proof. Let $\Pi := (t_0, t_1, ..., t_n) \in \mathcal{P}_{a,b}$. Then

$$\Lambda_\gamma(\Pi) = \sum_{i=1}^n ||\gamma(t_i) - \gamma(t_{i-1})|| = \sum_{i=1}^n || \int_{t_{i-1}}^{t_i} \gamma'(t)dt|| \le \sum_{i=1}^n \int_{t_{i-1}}^{t_i} ||\gamma'(t)||\, dt = \int_a^b ||\gamma'(t)||\, dt$$

In order to prove the Lemma, it remains to prove

$$\forall\, \varepsilon > 0\ \exists \Pi \in \mathcal{P}_{a,b} : \Big|\Lambda_\gamma(\Pi) - \int_a^b ||\gamma'(t)||\, dt\Big| < \varepsilon \qquad (7.)$$

Fix $\varepsilon > 0$. Recall that $||\gamma'|| \in L^1$, so that by a known density property, for every $j = 1, ..., N$ there exists α_j continuous such that $\int_a^b |\alpha_j - \gamma_j'| < \frac{\varepsilon}{3N}$ so that

$$\int_a^b ||\alpha - \gamma'|| \le \int_a^b \sum_{j=1}^N |\alpha_j - \gamma_j'| = \sum_{j=1}^N \int_a^b |\alpha_j - \gamma_j'| < \frac{\varepsilon}{3} . \qquad (7.)$$

By the uniform continuity there exists $\delta > 0$ such that

$$||\alpha(t) - \alpha(t')|| < \frac{\varepsilon}{6(b-a)} \text{ if } t, t' \in [a,b], |t - t'| < \delta$$

Let n be so big that $\frac{b-a}{n} < \delta$ and let $t_i = a + i\frac{b-a}{n}$, so that $\Pi = (t_0, t_1, ..., t_n) \in \mathcal{P}_{a,b}$, therefore $\Big|\Big| \int_{t_{i-1}}^{t_i} \alpha(t) - \alpha(t_{i-1})\Big|\Big| \le \int_{t_{i-1}}^{t_i} \Big|\Big|\alpha(t) - \alpha(t_{i-1})\Big|\Big| \le \frac{\varepsilon}{6(b-a)}(t_i - t_{i-1})$. We have

$$\Big|\, || \int_{t_{i-1}}^{t_i} \alpha(t)|| - ||(t_i - t_{i-1})\alpha(t_{i-1})||\, \Big| \le \Big|\Big| \int_{t_{i-1}}^{t_i} \alpha(t)\, dt\ - (t_i - t_{i-1})\alpha(t_{i-1})\Big|\Big| \le \frac{\varepsilon(t_i - t_{i-1})}{6(b-a)} ;$$

$$\Big| \int_{t_{i-1}}^{t_i} ||\alpha(t)|| - ||(t_i - t_{i-1})\alpha(t_{i-1})||\, \Big| \le \int_{t_{i-1}}^{t_i} \Big|\, ||\alpha(t)|| - || \int_{t_{i-1}}^{t_i} \alpha(t_{i-1})||\, \Big| \le$$

$$\int_{t_{i-1}}^{t_i} ||\alpha(t) - \alpha(t_{i-1})|| \le \frac{\varepsilon(t_i - t_{i-1})}{6(b-a)}$$

Therefore, $\Big| \, || \int_{t_{i-1}}^{t_i} \alpha(t)|| - \int_{t_{i-1}}^{t_i} ||\alpha(t)|| \, \Big| \le \frac{\varepsilon(t_i - t_{i-1})}{3(b-a)}$, hence

$$\Big| \int_a^b ||\alpha(t)|| - \sum_{i=1}^n || \int_{t_{i-1}}^{t_i} \alpha(t)|| \Big| =$$

$$\sum_{i=1}^n \Big| \int_{t_{i-1}}^{t_i} ||\alpha(t)|| - \sum_{i=1}^n || \int_{t_{i-1}}^{t_i} \alpha(t)|| \Big| \le$$

$$\sum_{i=1}^n \Big| \, || \int_{t_{i-1}}^{t_i} \alpha(t)|| - \int_{t_{i-1}}^{t_i} ||\alpha(t)|| \, \Big| \le \sum_{i=1}^n \frac{\varepsilon(t_i - t_{i-1})}{3(b-a)} = \frac{\varepsilon}{3}.$$

Moreover,

$$\Big| \int_a^b ||\gamma'(t)||dt - \int_a^b ||\alpha(t)||dt \Big| = \Big| \int_a^b ||\gamma'(t)|| - ||\alpha(t)||dt \Big|$$

$$\le \int_a^b \Big| \, ||\gamma'(t)|| - ||\alpha(t)|| \, \Big| dt \le \int_a^b ||\alpha - \gamma'|| < \frac{\varepsilon}{3};$$

$$\Big| \sum_{i=1}^n || \int_{t_{i-1}}^{t_i} \alpha(t)|| - \sum_{i=1}^n || \int_{t_{i-1}}^{t_i} \gamma'(t)|| \Big| \le \sum_{i=1}^n \Big| \, || \int_{t_{i-1}}^{t_i} \alpha(t)|| - || \int_{t_{i-1}}^{t_i} \gamma'(t)|| \, \Big| \le$$

$$\sum_{i=1}^n || \int_{t_{i-1}}^{t_i} \alpha(t) - \gamma'(t)|| \le \sum_{i=1}^n \int_{t_{i-1}}^{t_i} ||\alpha(t) - \gamma'(t)|| = \int_a^b ||\alpha(t) - \gamma'(t)|| < \frac{\varepsilon}{3}.$$

Now, (7.5) follows from the last three formulas. ∎

Note that if $\overline{\gamma} \in \mathcal{C}_{a,b,N}$ is a curve that minimizes the length on a class F of AC curves, $F \subseteq \mathcal{C}_{a,b,N}$, and is parametrized by (a multiple of) arclength, then it also minimizes the functional $I(\gamma) := \int_a^b ||\gamma'(t)||^2 \, dt$ on F. In fact, using the Jensen inequality we have

$$I(\overline{\gamma}) = \int_a^b ||\overline{\gamma}'(t)||^2 \, dt = \frac{1}{b-a} \Big(\int_a^b ||\overline{\gamma}'(t)|| \, dt \Big)^2 \le$$

$$\frac{1}{b-a}\Big(\int_a^b ||\gamma'(t)||\,dt\Big)^2 \leq \int_a^b ||\gamma'(t)||^2\,dt = I(\gamma)\,,$$

for every $\gamma \in F$, where I have used the fact that $||\overline{\gamma}'||$ is constant as $\overline{\gamma}$ is parametrized by a multiple of arclength.

Theorem 7.6. *If $\overline{\gamma}$ is a geodesics on a compact and connected surface S in $\mathbb{R}^3$ of class C^∞, parametrized by a multiple of arclength, then $\overline{\gamma}$ is of class C^1, and in fact also of class C^∞.*

Proof. Let $\overline{\gamma} : [a,b] \to S$, fix $\overline{t} \in [a,b]$, and put $\overline{P} = \overline{\gamma}(\overline{t})$. We using the notation of Theorem 4.2, where the open set V in $\mathbb{R}^2$ is choosed to be a ball centered at $(0,0)$ so small that

$$||grad\psi|| < \varepsilon \tag{7.7}$$

$$\Big|\frac{\partial^2\psi}{\partial y_i \partial y_j}\Big| \leq K \tag{7.8}$$

where ε is a sufficiently small positive number, and K is a suitable number. Such a number exists as $\frac{\partial^2\psi}{\partial y_i \partial y_j}$ are bounded on the compact $\overline{V}$. The function $(\overline{\gamma}_1, \overline{\gamma}_2)$ then minimizes the functional

$$J(y) := \int_c^d (y_1'(t))^2 + (y_2'(t))^2 + \Big(\frac{\partial\psi}{\partial y_1}(y(t))y_1'(t) + \frac{\partial\psi}{\partial y_2}(y(t))y_2'(t)\Big)^2$$

We have to study $L : [c,d] \times V \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$L(t,y,q) = q_1^2 + q_2^2 + \Big(\frac{\partial\psi}{\partial y_1}(y)q_1 + \frac{\partial\psi}{\partial y_2}(y)q_2\Big)^2$$

We take $\theta(r) = r^2$ so that we have

$$L(t,y,q) \geq ||q||^2 \tag{7.9}$$

Moreover, $\frac{\partial L}{\partial q_i}(t,y,q) = 2q_i + 2\Big(grad\psi(y)\cdot q\Big)\frac{\partial\psi}{\partial y_i}(y)$, hence

$$\Big|\Big|\frac{\partial L}{\partial q_i}(t,y,q)\Big|\Big| \leq A||q|| \leq A(1+\theta(||q||)) \tag{7.10}$$

where $A = 2 + 2\varepsilon^2$. Similarly, we get

$$\Big|\Big|\frac{\partial L}{\partial y_i}(t,y,q)\Big|\Big| \leq B||q|| \leq B(1+\theta(||q||)) \tag{7.11}$$

where $B = 4K\varepsilon$. Moreover,

$$\frac{\partial^2 L}{\partial q_i \partial q_j}(t,y,q) = 2\delta_{i,j} + 2\frac{\partial\psi}{\partial y_i}(y)\frac{\partial\psi}{\partial y_j}(y)$$

so that for suffciently small ε, the matrix $\frac{\partial^2 L}{\partial q_i \partial q_j}$ is positive definite, hence L is strictly convex with respect to q. Thus, in view of (7.9), (7.10), (7.11), the hypothesis of Theorem 6.4 is satisfied, and $(\overline{\gamma}_1, \overline{\gamma}_2)$, so $\overline{\gamma}$, is C^1, hence by a known theorem C^∞, on $[c,d]$ in particular at $\overline{t}$. As $\overline{t}$ is an arbitrary point in $[a,b]$, we have completed the proof. ∎