Picard bundles and syzygies of canonical curves

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To Professor Mario Fiorentini

Introduction. The purpose of this note is to show a relation between the syzygies of a curve completely embedded by its canonical bundle and the Ext-cohomology of Picard bundles on its Jacobian. The spirit of the result is to try to interpret geometrically Green's conditions N_p on the jacobian rather than on the curve. In order to describe the result, we need some preliminary material and notation:

(a) First order deformations of vector bundles on abelian varieties. Given a vector bundle F on an abelian variety X there are two obvious ways to deform it: the first one (as in any variety) is by tensoring it with a line bundle in Pic^0X and the second one is by translating it. So we have the two families of vector bundles $\{F \otimes \alpha\}_{\alpha \in \mathrm{Pic}^0 X}$ and $\{T_a^* F\}_{a \in X}$. Correspondingly, we have the Kodaira-Spencer maps

$$\Phi : H^1(\mathcal{O}_X) \to \mathrm{Ext}^1(F, F) \quad \text{and} \quad \Psi : T_{X,0} \to \mathrm{Ext}^1(F, F)$$

(where $T_{X,0}$ is the tangent space of X at the identity point 0).

(b) Picard bundles. Let C be a curve and L a line bundle of degree $d \geq 2g(C) - 1$ on C. Let $\mathcal{P}$ be a Poincaré line bundle on $C \times \mathrm{Pic}^0 C$ and let p and q the projections. For us a *Picard bundle* will be a vector bundle on $\mathrm{Pic}^0 C$ of the form

$$\mathcal{E}_L = q_*(p^*(L) \otimes \mathcal{P}).$$

By Riemann-Roch and Grauert's theorem, if $\alpha \in \mathrm{Pic}^0 C$ parametrizes a line bundle of degree zero (which, by abuse of language, we will call α too), the fibre $\mathcal{E}_L(\alpha)$ is $H^0(L \otimes \alpha)$. We refer to [Sc],[K1],[Mu],[EL] and references therein for basic results about these bundles. In particular Kempf [K1] and Mukai [Mu] have shown that the curve C is non-hyperelliptic if and only if the map

$$\Phi \oplus \Psi : H^1(\mathcal{O}_{\mathrm{Pic}^0 C}) \oplus T_{\mathrm{Pic}^0 C, 0} \to \mathrm{Ext}^1(\mathcal{E}_L, \mathcal{E}_L)$$

is an isomorphism, i.e. at the first order level, the space of all deformations of $\mathcal{E}_L$ is the product of the two families above. The reader is referred to [K1] and [Mu] for more on the geometric meaning of this result (compare also the end of Section 4 below).

(c) Higher Ext*'s.* Returning to (a), it turns out that $\mathrm{Ext}^\bullet(F, F)$ is naturally a graded module over the cohomology algebra $H^\bullet(\mathcal{O}_X)$ and the two maps Φ and Ψ are the degree-one pieces of maps of graded modules

$$\Phi^\bullet : H^\bullet(\mathcal{O}_X) \to \mathrm{Ext}^\bullet(F, F) \quad \text{and} \quad \Psi^\bullet : T_{X,0} \otimes H^{\bullet - 1}(\mathcal{O}_X) \to \mathrm{Ext}^\bullet(F, F)$$

This is shown in Section 1 below.

(d) Syzygies of canonical curves. Let $R = \bigoplus_{i\geq 0} H^0(K^{\otimes i})$ be the canonical ring of C. It is naturally a graded module over the polynomial ring $S = \bigoplus_{i\geq 0} \mathrm{Sym}^i(H^0(K))$. Let us consider a minimal resolution of R as graded S-module

$$L^\bullet \qquad 0 \to E_{g-2} \to E_{g-1} \to \cdots \to E_1 \to E_0 \to R \to 0$$

where

$$E^j = \bigoplus S(-k)^{\oplus b_{jk}}.$$

The Betti numbers b_{jk} are intrinsic invariants of C so they must be related to the intrinsic geometry of C. A precise relation of this type has been conjectured by Green ([G]). To state it let us recall some terminology: one says that C satisfies property N_0 if $E_0 = S$ (i.e. $b_{0k} = 0$ for $k \neq 0$ and $b_{00} = 1$). If K is very ample, this means that the canonical curve is projectively normal. Next, one says that C satiesfies property N_1 if it satisfies N_0 and $b_{1k} = 0$ for $k \neq 2$. This means that the homogeneous ideal of the canonical curve is generated by quadrics. Inductively, C is said to satisfy N_p if it satisfies N_{p-1} and moreover $b_{pk} = 0$ for $k \neq p+1$. In a word, C satisfies N_p if the resolution $L^\bullet$ is linear up to the p-th step. In this language Noether's theorem states that C satisfies N_0 if and only if it is not hyperelliptic, while Petri's theorem states that if C satisfies N_0, then it satisfies N_1 if and only if it is not trigonal or isomorphic to a plane quintic. Green conjectures is that C satisfies N_p if and only if the Clifford index of C is stricly greater than p, a statement recovering the cases $p = 0, 1$. We refer to [G],[L1],[S],[V1],[V2] for the definition of Clifford index, discussions and results.

Our purpose here is not to add any evidence to the truth or not of Green's conjecture, but just to make a remark about the nature of conditions N_p. The result is

Theorem A. *Let $\mathcal{E}$ be any Picard bundle on $\mathrm{Pic}^0 C$. Then C satisfies property N_p if and only if the map of graded $H^\bullet(\mathcal{O}_{\mathrm{Pic}^0 C})$-modules*

$$\Phi^\bullet \oplus \Psi^\bullet : H^\bullet(\mathcal{O}_{\mathrm{Pic}^0 C}) \oplus (T_{\mathrm{Pic}^0 C,0} \otimes H^{\bullet-1}(\mathcal{O}_{\mathrm{Pic}^0 C})) \to \mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E})$$

is surjective in any degree k such that $1 \leq k \leq p+1$.

Actually, a variant of the statement will follow from the proof of Theorem A. More precisely it will follow that condition N_p holds if and only if $\mathrm{Ext}^k(\mathcal{E}, \mathcal{E})$ have the expected dimension for any k such that $1 \leq k \leq p+1$. We refer to the beginning of section 4 for details. Since $\mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E})$ is self-dual, this implies also that $\mathrm{Ext}^k(\mathcal{E}, \mathcal{E})$ has the expected dimension also for any k such that g–p–$1 \leq k \leq g$–1. In particular, for *general* curves of genus g, Green's conjecture translates to the statement that all the graded components of $\mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E})$ have the minimal dimension. It is worth to remark (see [K1] and [Mu], compare also Theorem 2.1 below) that in any case $k \cong H^0(\mathcal{O}_{\mathrm{Pic}^0 C}) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{E})$ (i.e. $\mathcal{E}$ is *simple*) and the map $\Phi^\bullet \oplus \Psi^\bullet$ is always injective in degree 1.

To put the result into perspective, let us recall that in the papers [K1] and [K2] are proved two results, somehow complementary, on the first-order deformations respectively of of symmetric products $C^{(d)}$ and of Picard bundles $\mathcal{E}$. The first one deals with the derivative $du : T_{C^{(d)}} \to u^*T_{J(C)}$ of the Abel-Jacobi map $u : C^{(d)} \to J(C)$ and states, in particular, that $H^1(du)$ is injective if and only if C satisfies N_0. This have been generalized to higher syzygies by Lazarsfeld in [L2]. The second one is the result about first order deformations of Picard bundles mentioned in (b) above (proved independentey also by Mukai in [Mu]). Therefore Theorem A can be seen as the analogue for Picard bundles of Lazarsfeld's generalization. (It should also be mentioned that also a part of the statement of Theorem A in degree two can be found in Kempf's paper ([K1], Theorem 6.7).

We will work over an algebraically closed field k.

The debt of the present paper to Kempf's article [1] does not need to be acknowledged. This note is dedicated to Professor Mario Fiorentini with deep friendship and gratitude.

1. The maps $\Phi^\bullet$ and $\Psi^\bullet$. Let F be a locally free sheaf an abelian variety X.

(a) There is the canonical isomorphism

$$H^j(\mathcal{H}om(F,F)) \cong \mathrm{Ext}^j(F,F). \tag{1.1}$$

Thus $\mathrm{Ext}^j(F,F)$ is naturally a graded $H^\bullet(\mathcal{O}_X)$-module via cup product. Explicitely, the map

$$\Phi^\bullet : H^\bullet(\mathcal{O}_X) \to \mathrm{Ext}^\bullet(F,F) \tag{1.2}$$

is defined considering the omothety map $\mathcal{O}_X \to \mathcal{H}om(F,F)$ and taking cohomology. Of course $H^\bullet(\mathcal{O}_X)$ is the exterior algebra $\Lambda^\bullet H^1(\mathcal{O}_X)$. As we said, the degree-one map Φ^1 is the Kodaira-Spencer map of the family of $\{F \otimes \beta\}_{\beta \in \mathrm{Pic}^0 X}$.

(b) We have a another map of graded-$H^\bullet(\mathcal{O}_X)$-modules

$$\Psi^\bullet : T_{X,0} \otimes H^{\bullet-1}(\mathcal{O}_X) \to \mathrm{Ext}^\bullet(F,F) \tag{1.3}$$

($T_{X,0}$ is the tangent space of X at 0) defined as follows. One has the bundle of principal parts of F: $P^1(F) = p_{1*}(p_2^*F)_{|\Delta^{(2)}})$ (p_i are the projections on $X \times X$ and $\Delta^{(2)}$ is the first infinitesimal neighborhood of the diagonal in $X \times X$). It sits in the canonical extension

$$0 \to E \otimes \Omega^1_X \to P^1(E) \to E \to 0 \tag{1.4}$$

Applying $\mathcal{H}om(F,\cdot)$ and using that Ω^1_X is trivial and isomorphic to $\Omega^1_{X,0} \otimes \mathcal{O}_X$, one gets

$$0 \to \Omega^1_{X,0} \otimes \mathcal{H}om(F,F) \to \mathcal{H}om(F,P^1(F)) \to \mathcal{H}om(F,F) \to 0 \tag{1.5}$$

The coboundaries of the short exact sequence (1.5) give a map $H^{\bullet-1}(\mathcal{H}om(F,F)) \to T^\vee_{0,X} \otimes H^\bullet(\mathcal{H}om(F,F))$. Composing with (1.2) and contracting one gets $\Psi^\bullet$. As shown in Sect.8 of [K1], the degree-one map Ψ^1 is the Kodaira-Spencer map of the family $\{T_x^*F\}_{x \in X}$, where T_x denotes the traslation $y \mapsto y + x$.

2. Computing $\mathbf{Ext}^\bullet(\mathcal{E}_\mathbf{L},\mathcal{E}_\mathbf{L})$. Let K be the canonical bundle of C and let us consider the bundle M_K, defined by the exact sequence

$$0 \to M_K \to H^0(K)\otimes\mathcal{O}_C \to K \to 0 \tag{2.1}$$

Here is the basic computation of the paper:

Theorem 2.1. *For any $i \geq 0$ there is a canonical exact sequence*

$$0 \to H^1(\overset{p-1}{\Lambda} M_K^\vee) \to \mathrm{Ext}^p(\mathcal{E},\mathcal{E}) \to H^0(\overset{p}{\Lambda} M_K^\vee) \to 0 \tag{2.2}$$

Proof. The method of proof is the one of Kempf's work [K1]. Here we refine Kempf's computations using the general construction of the article [P], where these ideas are applied to a totally different context.

STEP 1. We will work on $C\times C\times \mathrm{Pic}^0C$. Let us denote p_i the three projections and p_{ij} the intermediate projections. Moreover let $\mathcal{P}$ be a Poincaré line bundle on $C\times\mathrm{Pic}^0C$ and Δ the diagonal in $C\times C$. For any integer k let us consider on $C\times C\times\mathrm{Pic}^0C$ the line bundles

$$M^k_{L^+,K\otimes L^{\vee-}} = p_{13}^*(p_1^*(L)\otimes\mathcal{P})\otimes p_{23}^*(p_2^*(K\otimes L)\otimes\mathcal{P}^\vee))\otimes p_{12}^*(\mathcal{O}_{C\times C}(-k\Delta)) \tag{2.3}$$

where $\mathcal{P}$ is a Poincaré bundle on $C\times X$. We have that

$$H^1(M^0_{L^+,K\otimes L^{\vee-}}{}_{|C\times C\times\{\alpha\}}) = H^0(L\otimes\alpha)\otimes H^1(K\otimes L^\vee\otimes\alpha^\vee)$$

which is, by Serre duality, isomorphic to $\mathrm{Hom}(H^0(L\otimes\alpha),H^0(L\otimes\alpha))$. On the other hand, for any $\alpha\in\mathrm{Pic}^0C$ and $i=0,2$, we have that $H^i(M^0_{L^+,K\otimes L^{\vee-}}{}_{|C\times C\times\{\alpha\}})=0$ (recall that $\deg(L)\geq 2g(C)-1$). Therefore, by relative duality and Grauert's theorem, we have that $R^ip_{3*}M^0_{L^+,K\otimes L^{\vee-}} \cong \mathcal{H}om(\mathcal{E},\mathcal{E})$ if $i=1$ and zero otherwise. Thus the Leray spectral sequence of p_3 degenerates giving isomorphisms

$$H^j(\mathcal{H}om(\mathcal{E},\mathcal{E})) \cong H^{j+1}(M^0_{L^+,K\otimes L^{\vee-}}) \tag{2.4}$$

Hence we are reduced to compute the cohomology of $M^0_{L^+,K\otimes L^{\vee-}}$. To do that we will apply the projection p_{12} onto $C\times C$ and study the Leray spectral sequence.

STEP 2. For future reference, let us first record the following key results about duality between abelian varieties

Theorem 2.2. ([M], p.127) *Let A be an abelian variety of dimension q, Pic^0A the dual variety and $\mathcal{Q}$ a Poincaré line bundle. Then*

$$R^kp_{A*}\mathcal{Q} = \begin{cases} H^q(\mathcal{O}_{\mathrm{Pic}^0A})\otimes\mathcal{O}_0 & \text{for } k=q \\ 0 \quad \text{otherwise} \end{cases}$$

(where $\mathcal{O}_0$ is the one-dimensional skyscraper sheaf on the point 0 of A).

Corollary 2.3. ([K1] Cor.2.2) *Let T be a variety and $\pi : T \to A$ a morphism. Then we have functorial isomorphisms*

$$R^i p_{T*}((\pi \times id_{\mathrm{Pic}^0 A})^*(\mathcal{Q})) \cong \mathcal{T}or_{q-i}^{\mathcal{O}_A}(\mathcal{O}_T, H^q(\mathcal{O}_{\mathrm{Pic}^0 A}) \otimes \mathcal{O}_0)$$

(where $\mathcal{O}_T$ is seen as $\mathcal{O}_A$-module via π.)

Corollary 2.3 can be applied to our line bundle $M^k_{L^+, K\otimes L^\vee -}$ on $C \times C \times \mathrm{Pic}^0 C$ as follows: let us take $A = \mathrm{Alb}C$, $T = C \times C$ and $\pi = d \circ (a \times a)$, where $d : A \times A \to A$ is the difference map $(x, y) \to x - y$ and $a : C \to A$ is a fixed Albanese (Abel-Jacobi) map. Since $\mathrm{Pic}^0 C = \mathrm{Pic}^0 A$ and the Poincaré bundle $\mathcal{P}$ on $C \times \mathrm{Pic}^0 C$ is the pullback via a of a Poincaré bundle $\mathcal{Q}$ on $A \times \mathrm{Pic}^0 A$, i.e. $\mathcal{P} = (a, id_{\mathrm{Pic}^0 A})^*(\mathcal{Q})$, it is easily seen that, by the Theorem of the cube ([M] p.91),

$$p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}^\vee \cong (\pi \times id_{A^\vee})^*\mathcal{Q} \tag{2.5}$$

Therefore

$$M^k_{L^+, K\otimes L^\vee -} = p_1^*(L) \otimes p_2^*(K \otimes L^\vee) \otimes p_{12}{}^*(\mathcal{O}_{C\times C}(-k\Delta)) \otimes (\pi \times id_{\mathrm{Pic}^0 A})^*(\mathcal{Q}) \tag{2.6}$$

Applying Cor.2.2 and projection formula we get

$$R^i p_{12*} M^k_{L^+, K\otimes L^\vee -} \cong \mathcal{T}or_{q-i}^{\mathcal{O}_A}(\mathcal{O}_{C\times C}, \mathcal{O}_0) \otimes p_1^*(L) \otimes p_2^*(K \otimes L^\vee) \otimes p_{12}^*(\mathcal{O}_{C\times C}(-k\Delta)) \tag{2.7}$$

where $\mathcal{O}_{C\times C}$ is seen as an $\mathcal{O}_A$-module via π and we have made, in order to make the notation less heavy, the identification $H^q(\mathcal{O}_{\mathrm{Pic}^0 C}) \cong k$. So (2.7) yields that the sheaves $R^h p_{12*} M^k_{L^+, K\otimes L^\vee -}$ are supported on the diagonal $\Delta = \pi^{-1}(0)$ and in fact

$$R^i p_{12*} M^k_{L^+, K\otimes L^\vee -} \cong \mathcal{T}or_{q-i}^{\mathcal{O}_A}(\mathcal{O}_{C\times C}, \mathcal{O}_0) \otimes K^{\otimes k+1} \tag{2.8}$$

In particular, their cohomology vanishes for $j \geq 2$. Therefore we have achieved a first result: the spectral sequence $H^j(R^h p_{12*} M^k_{L^+, K\otimes L^\vee -}) \Rightarrow H^{j+h}(M^k_{L^+, K\otimes L^\vee -})$ degenerates as follows

$$\begin{gathered} \cdots \xrightarrow{\delta^k_{j-1}} H^1(R^{j-1} p_{12*} M^k_{L^+, K\otimes L^\vee -}) \to H^j(M^k_{L^+, K\otimes L^\vee -}) \to H^0(R^j p_{12*} M^k_{L^+, K\otimes L^\vee -}) \xrightarrow{\delta^k_j} \\ \xrightarrow{\delta_j} H^1(R^j p_{12*} M^k_{L^+, K\otimes L^\vee -}) \to H^{j+1}(M^k_{L^+, K\otimes L^\vee -}) \to H^0(R^{j+1} p_{12*} M^k_{L^+, K\otimes L^\vee -}) \xrightarrow{\delta_{j+1}} \end{gathered} \tag{2.9}$$

Then Theorem 2.1 will follow from the following

Claim. *(i) $R^j p_{12*} M^0_{L^+, K\otimes L^\vee -} = \Lambda^{j-1} M_K^\vee$; (ii) the connecting maps δ^k_j in (2.9) are zero.*

STEP 3. Here we will prove (i) of the Claim. Let $\tilde{\Delta} = \Delta \times \mathrm{Pic}^0 C$. Since $p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}^\vee_{|\tilde{\Delta}}$ is trivial and $(p_{13}^* L \otimes p_{23}^K \otimes L^\vee)(-k\Delta))_\Delta = K^{\otimes k+1}$ we have the basic sequences

$$0 \to M^{k+1}_{L^+, K\otimes L^\vee -} \xrightarrow{\cdot\tilde{\Delta}} M^k_{L^+, K\otimes L^\vee -} \xrightarrow{\rho^k} p_{12}^*(\Delta_*(K^{\otimes k+1})) \to 0 \tag{2.10}$$

where Δ means here the diagonal embedding of C into $C \times C$. Applying p_{12*} one gets a long cohomology sequence

$$\cdots \xrightarrow{\theta_k^i} R^{i-1}p_{12*}M^k_{L^+,K\otimes L^\vee -} \to \Lambda^i H^0(K)^\vee \otimes \Delta_*(K^{\otimes k+1}) \to R^i p_{12*}M^{k+1}_{L^+,K\otimes L^\vee -} \xrightarrow{\theta_k^{i+1}} \cdots \tag{2.11$_{i,k}$}$$

(recall that – by duality of abelian varieties – we have that $H^1(\mathcal{O}_{\mathrm{Pic}^0 X}) \cong T_{\mathrm{Pic}^0(\mathrm{Pic}^0 X),0} \cong T_{\mathrm{Alb}X,0} \cong H^0(K)^\vee$). Applying the (functorial) isomorphism (2.7) to sequences (2.11) we get the long homology sequence of $\mathcal{T}or$'s

$$\begin{aligned} \cdots \to \mathcal{T}or^{\mathcal{O}_A}_{q-i}(\mathcal{O}_{C\times C}, \mathcal{O}_0) \otimes K^{\otimes k+1} \to \mathcal{T}or^{\mathcal{O}_A}_{q-i}(\mathcal{O}_\Delta, \mathcal{O}_0) \otimes K^{\otimes k+1} \to \\ \to \mathcal{T}or^{\mathcal{O}_A}_{q-i-1}(\mathcal{O}_{C\times C}, \mathcal{O}_0) \otimes K^{\otimes k+2} \to \cdots \end{aligned} \tag{2.12}$$

associated to the sequence $0 \to \mathcal{O}_{C\times C}(-(k+1)\Delta) \xrightarrow{\cdot\Delta} \mathcal{O}_{C\times C}(-k\Delta) \to K^{\otimes k} \to 0$, tensored with K. Now, since all sheaves are supported on Δ and since the maps $\mathcal{T}or^{\mathcal{O}_A}_h(\cdot\Delta, \mathcal{O}_0)$ are zero when restricted to Δ, the long sequences (2.11) are in fact chopped into short sequences

$$0 \to R^{i-1}p_{12*}M^k_{L^+,K\otimes L^\vee -} \to \overset{q-i+1}{\Lambda} H^0(K) \otimes K^{\otimes k+1} \to R^i p_{12*}M^{k+1}_{L^+,K\otimes L^\vee -} \to 0. \tag{2.13}$$

Therefore for $i = q$ we get

$$R^i p_{12*}M^k_{L^+,K\otimes L^\vee -} \cong K^{\otimes k+1}$$

proving (i) of the Claim for $i = q$ (since the determinant $\Lambda^{g-1}M_K^\vee$ is equal to K). Next, considering $i = q-1$, we have the exact sequence

$$0 \to R^{q-1}p_{12*}M^k_{L^+,K\otimes L^\vee -} \to H^0(K) \otimes K^{\otimes k+1} \to K^{\otimes k+2} \to 0 \tag{2.14}$$

and, via the isomorphism (2.7), the third arrow in (2.14) is the evaluation map $H^0(K) \to K$ tensored with $K^{\otimes k}$. Thus

$$R^{q-1}p_{12*}M^k_{L^+,K\otimes L^\vee -} \cong M_K \otimes K^{\otimes k+1} \cong \overset{q-2}{\Lambda} M_K^\vee \otimes K^{\otimes k} \tag{2.15}$$

where the last isomorphism follows from duality in the exterior algebra and (2.14) is identified to the exact sequence

$$0 \to M_K \otimes K^{\otimes k+1} \to H^0(K) \otimes K^{\otimes k+1} \to K^{\otimes k+2} \to 0$$

i.e. our basic sequence (2.1) twisted by $K^{\otimes k+1}$. Arguing inductively one gets in a similar fashion (compare [K1] Section 6) that

$$R^i p_{12*}M^k_{L^+,K\otimes L^\vee -} \cong \overset{q-i}{\Lambda} M_K \otimes K^{\otimes k+1} \cong \Lambda^{i-1}M_K^\vee \otimes K^{\otimes k} \tag{2.16}$$

and the exact sequences (2.13) are identified with the exact sequences:

$$0 \to \overset{q-i}{\Lambda} M_K \otimes K^{\otimes k+1} \to \overset{q-i}{\Lambda} H^0(K) \otimes K^{\otimes k+2} \to \overset{q-i+1}{\Lambda} K^{\otimes k+2} \to 0 \tag{2.17}$$

obtained taking exterior products in (2.1) and tensoring with $K^{\otimes k}$. Thus we have proved (i) of the Claim.

STEP 4. Here we will prove (ii) of the Claim, which will conclude the proof of Theorem 2.1. The proof of (ii) of the Claim will be by descending induction on k. If k is high enough , so that $H^1(\Lambda^{q-i} M_K \otimes K^{\otimes k+1}) = 0$ for any j, (actually it can be shown that $k = 1$ suffices) the claim is obvious. To prove the induction step, note that, since the long cohomology sequences $(2.11)_{i,k}$ are chopped into the short exact sequences (2.13) (i.e., written in a different way, (2.17)), the corresponding hypercohomologies degenerate fitting in a commutative exact diagram

$$\begin{array}{ccccccc}
 & & & & & & \downarrow 0 & \\
\xrightarrow{\delta^k_{i-1}} & H^1(R^{i-1}_{p_{12}\,*} M^k_{L^+,K\otimes L^\vee -}) & \to & H^i(M^k_{L^+,K\otimes L^\vee -}) & \to & H^0((R^i_{p_{12}*} M^k_{L^+,K\otimes L^\vee -})) & \xrightarrow{\delta^k_i} \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^1(R^{i-1}_{p_{12}\,*}(M^k_{L^+,K\otimes L^\vee -}{}_{|\tilde{\Delta}})) & \to & H^i(M^k_{L^+,K\otimes L^\vee -}{}_{|\tilde{\Delta}}) & \to & H^0(R^i_{p_{12}*}(M^k_{L^+,K\otimes L^\vee -}{}_{|\tilde{\Delta}})) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \xrightarrow{\delta^{k+1}_i} & H^1(R^i_{p_{12}*} M^{k+1}_{L^+,K\otimes L^\vee -}) & \to & H^{i+1}(M^{k+1}_{L^+,K\otimes L^\vee -}) & \to & H^0(R^{i+1}_{p_{12}\,*} M^{k+1}_{L^+,K\otimes L^\vee -}) & \xrightarrow{\delta^{k+1}_{i+1}} 0 \\
 & \downarrow 0 & & & & &
\end{array}$$

where:
(a) the middle horizontal short exact sequence is given by Künneth formula (recall that $M^k_{L^+,K\otimes L^{\vee\vee}} = p^*_{12}(\Delta_*(K^{\otimes k+1}))$);
(b) the third horizontal sequence is exact by induction;
(c) the middle vertical exact sequence is the long cohomology sequence of (2.10),
(d) the left and right vertical exact sequence are the cohomology sequences of (2.13) (or, equivalently, of (2.17)).

Then, by the snake lemma $\delta^k_i = 0$. Therefore $\delta^k_i = 0$ for any i and k. □

3. Syzygies of canonical curves. Here we will recall some more basic facts – largely due to Green – about syzygies of canonical curves (again we refer to [G] and [L1] for details). We keep the notation of (d) in the Introduction:

(a) it is easy to see that in any case $b_{pk} = 0$ for $k \neq p+1, p+2$, i.e. at each step condition N_p can fail af most by one. This follows e.g. from Castelnuovo-Mumford regularity.

(b) (see e.g. [L1] p.511). We have that $b_{p,p+2} = 0$ (i.e. N_p holds) if and only if the following complex $\mathcal{K}^\bullet_p$ is exact in the middle

$$\mathcal{K}^\bullet_p : \quad \overset{p+1}{\Lambda} H^0(K) \otimes H^0(K) \to \overset{p}{\Lambda} H^0(K) \otimes H^0(K^{\otimes 2}) \to \overset{p-1}{\Lambda} H^0(K) \otimes H^0(K^{\otimes 3})$$

(c) Taking wedge products of sequence (2.1) we get

$$0 \to \overset{p+1}{\Lambda} M_K \otimes K \to \overset{p+1}{\Lambda} H^0(K) \otimes K \to \overset{p}{\Lambda} M_K \otimes K^{\otimes 2} \to 0. \tag{3.1}$$

It is easy to see that the fact that the complex $\mathcal{K}_p^\bullet$ is exact means that the map

$$\overset{p+1}{\Lambda} H^0(K) \otimes H^0(K) \to H^0(\overset{p}{\Lambda} M_K \otimes K^{\otimes 2}) \tag{3.2}$$

is surjective, and this is in turn equivalent to $h^1(\Lambda^{p+1} M_K \otimes K) = \binom{g}{p+1}$ (since, as it is easy to see, (a) yields that $H^1(\Lambda^p M_K \otimes K^{\otimes 2}) = 0$ for any j). Actually, it will more convenient for us the dual version. Applying $\mathcal{H}om(\cdot, K)$ to (3.1) one gets

$$0 \to \overset{p}{\Lambda} M_K^\vee \otimes K^\vee \to \overset{p+1}{\Lambda} H^0(K)^\vee \otimes \mathcal{O}_C \to \overset{p+1}{\Lambda} M_K^\vee \to 0 \tag{3.3}$$

Then N_p holds if and only if the map $H^1(\Lambda^p M_K^\vee \otimes K^\vee) \to \Lambda^{p+1} H^0(K)^\vee \otimes H^1(\mathcal{O}_C)$ is injective. This is equivalent to the fact that the injective map $\Lambda^{p+1} H^0(K)^\vee \to H^0(\Lambda^{p+1} M_K^\vee)$ is an isomorphism i.e. $h^0(\Lambda^{p+1} M_K^\vee) = \binom{g}{p+1}$. So N_p means that $H^0(\Lambda^{p+1} M_K^\vee)$ has the minimal possible dimension.

4. Proof of Theorem A. Proposition 2.1 shows already that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = k$ and that $\mathrm{Ext}^p(\mathcal{E}, \mathcal{E})$ has the expected dimension if and only if $H^1(\Lambda^{p-1} M_K^\vee)$ and $H^0(\Lambda^p M_K^\vee)$ do, that is, by Section 3, if and only if C satisfies N_{p-1} and N_p. Moreover, since $N_p \Rightarrow N_{p-1}$, we can summarize the above remarks as follows: in any case

$$\dim(\mathrm{Ext}^p(\mathcal{E}, \mathcal{E}) \geq \binom{g}{p-1} - \chi(\overset{p-1}{\Lambda} M_K^\vee) + \binom{g}{p}$$

and we have equality if and only if C satisfies condition N_p.

To recover the statement of Theorem A, a few remarks are in order. First of all, by Theorem 2.1 we have the exact sequence

$$0 \to H^1(\overset{\bullet-1}{\Lambda} M_K^\vee) \to \mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E}) \to H^0(\overset{\bullet}{\Lambda} M_K^\vee) \to 0 \tag{4.1}$$

Let us recall that we have a canonical identification

$$H^1(\mathcal{O}_C) \cong T_{0, \mathrm{Pic}^0 C} \tag{4.2}$$

and a canonical identification of algebras

$$\overset{\bullet}{\Lambda} H^0(K)^\vee \cong H^\bullet(\mathcal{O}_{\mathrm{Pic}^0 C}) \tag{4.3}$$

(given by duality: $\mathrm{Pic}^0(\mathrm{Pic}^0 C) = \mathrm{Alb} C$). Then considering the H^0 of the third arrow in sequences (3.3) and using (4.3) one gets a map

$$H^\bullet(\mathcal{O}_{\mathrm{Pic}^0 C}) \to H^0(\overset{\bullet}{\Lambda} M_K^\vee), \tag{4.4}$$

while considering the H^1 one gets a map

$$T_{\mathrm{Pic}^0 C,0} \otimes H^{\bullet-1}(\mathcal{O}_{\mathrm{Pic}^0 C}) \to H^1(\overset{\bullet-1}{\Lambda} M_K^\vee) \tag{4.5}$$

(it can be seen that they are in fact maps of graded $H^\bullet(\mathcal{O}_{\mathrm{Pic}^0 C})$-modules). To conclude the proof of Theorem A we need only to show the folloing claim: via sequence (4.1), the map (4.4) lifts to $\Phi^\bullet$ and the map (4.5) is $\Psi^\bullet$. In fact, assuming the claim true, since (4.5) is always surjective, N_p holds if and only if $\Phi^\bullet$ is surjective (i.e. an isomorphism) in degree $\leq p+1$ i.e. if and only if $\Phi^\bullet \oplus \Psi^\bullet$ is surjective in degree $\leq p+1$. (Note that in this case $\Phi^\bullet$ gives a splitting of (4.1) up to degree $p+1$.) To prove that (4.4) lifts to the map $\Phi^\bullet$ is easy: this amounts to say that the third arrow of (4.1) is a map of graded $H^\bullet(\mathcal{O}_{\mathrm{Pic}^0 C})$-modules, where $\mathrm{Ext}^\bullet(\mathcal{E},\mathcal{E})$ has the module structure given by the map $\Phi^\bullet$ and $H^0(\Lambda^\bullet M_K^\vee)$ is equipped by the module structure given by (4.4) itself, and this follows immediately from the way Theorem 2.1 was proved.

It remains to verify that (4.5) is – via the injection of (4.1) – really the map $\Psi^\bullet$. First of all let us note that it is enough to prove that the two maps coincide in degree 1: indeed, by construction, both (4.5) and $\Psi^\bullet$ are obtained from the respective maps in degree 1 by composition with cup product:

$$\begin{array}{ccc} T_{0,\mathrm{Pic}^0 C} \otimes H^{j-1}(\mathcal{O}_{\mathrm{Pic}^0 C}) & \to & \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \otimes H^{j-1}(\mathcal{O}_{\mathrm{Pic}^0 C}) \\ & & \downarrow \\ & & \mathrm{Ext}^j(\mathcal{E},\mathcal{E}) \end{array}$$

The fact that $\Psi^\bullet$ and (4.5) are the same map in degree 1 is proved in [K1] (Prop.8.3). A perhaps geometrically more natural proof, although less homogeneous with the methods and notation of this paper, follows from Mukai's work [Mu]. Let us quickly outline it: the point is to show that via the canonical isomorphism $H^1(\mathcal{O}_C) \cong T_{0,\mathrm{Pic}^0 C}$, the degree-one map in (4.5) is the Kodaira-Spencer map of the family $\{\mathcal{E}_{L\otimes\alpha}\}_{\alpha\in\mathrm{Pic}^0 C}$. This is what we want, since by construction $\mathcal{E}_{L\otimes\alpha} = T^*_{\{\alpha\}}\mathcal{E}_L$ and $\Psi^\bullet$ is the Kodaira-Spencer map of the family of translations $\{T^*_{\{\alpha\}}\mathcal{E}\}_{\alpha\in\mathrm{Pic}^0 C}$. To prove what claimed, one uses the Mukai-Fourier transform: choosing an Abel-Jacobi embedding a of C in Alb$C$ one can see L as a sheaf on Alb$C$ via a. Then the "Fourier functor" induces a natural map

$$\mathrm{Ext}^p_{\mathcal{O}_{\mathrm{Alb}C}}(L,L) \to \mathrm{Ext}^p_{\mathcal{O}_{\mathrm{Pic}^0 C}}(\mathcal{E}_L,\mathcal{E}_L) \tag{4.6}$$

which turns out to be an isomorphism ([Mu] Cor.2.5) for any p. Plugging such an isomorphism for $p=1$ into the beginning of the spectral sequence of local-to-global Ext one has

$$0 \to H^1(\mathcal{O}_C) \to \mathrm{Ext}^1(\mathcal{E}_L,\mathcal{E}_L) \to H^0(\mathcal{N}_{C|\mathrm{Alb}C}) \to 0 \tag{4.7}$$

where $\mathcal{N}$ means normal bundle (compare [Mu], Proof of Lemma 4.9). (Note that this is another way of finding the degenerate spectral sequence (2.9)). The deformation theoretic meaning of (4.7) is well known: the injection is the Kodaira-Spencer map of our claim, and corresponds to deforming the line bundle L on the curve C, while the term $H^0(\mathcal{N}_{C|\mathrm{Alb}C})$

is the tangent space to the Hilbert scheme of C in Alb$C$ and corresponds to moving the curve inside Alb$C$. Going trough the ways (4.5) for $j=1$ and (4.7) are obtained, it is easy to show that they are the *same* exact sequence (note that $M_K^\vee = \mathcal{N}_{C|\mathrm{Alb}C}$) and this proves what claimed. □

References.

[EL] Ein, L., Lazarsfeld, R.: *Stability and restrictions of Picard bundles, with an application to the normal bundles of elliptic curves*, in Complex Projective Geometry (Ellingsrud, G., et al. eds), Cambridge Univ. Press (1992) 149-157

[G] Green, M. *Koszul cohomology and the geometry of projective varieties*, J. Diff. Geom. 19 (1984) 125-171

[K1] Kempf, G.: *Towards the inversion of abelian integrals,I* Ann. of Math. 110 (1979) 243-273

[K2] Kempf, G.: *Deformations of symmetric products*, in Riemann surfaces and related topics (Kra,I., Maskit, B. eds) Princeton Univ. Press (1981) 343-369

[L1] Lazarsfeld, R.: *A sampling of vector bundles techinques in the study of linear series* in Riemann surfaces (Cornalba, M. et al. eds) World Scientific (1989)

[L2] Lazarsfeld, R.: *Cohomology on symmetric products, syzygies of canonical curves and a theorem of Kempf*, in Einstein metrics and Yang-Mills connections (Mabuchi, T. and Mukai, S. eds) Marcel Dekker (1993) 89-90

[M] Mumford, D.: Abelian varieties, Second edition, Oxford Univ. Press (1974)

[Mu] Mukai, S.: *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves* , Nagoya math.J. 81 (1981) 153-175

[P] Pareschi, G. *Gaussian maps, hyperplane sections and generic vanishing of the H^1*, preprint

[S] Schreyer, F.: *A standard basis approach to syzygies of canonical curves*, J. reine angew. Math. 421 (1991) 83-123

[Sc] Schwarzenberger, R.: *Jacobian and symmetric products*, Ill. J. Math. 7 (1963), 257-268

[V1] Voisin, C.: *Courbes tétragonales et cohomology de Koszul*, J. reine angew. Math. 387 (1988) 111-121

[V2] Voisin, C.: *Deformation des syzygies et theorie de Brill-Noether*, Proc. London Math. Soc. 67 (1993) 493-515

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