L. A. G. test n. 7, june 17, 2016 Name:

1. For $t \in \mathbf{R}$ let $A_t = \begin{pmatrix} -5 & t-1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{pmatrix}$.

(a) For which $t \in \mathbf{R}$ is the matrix A_t diagonalizable?
(b) For all $t \in \mathbf{R}$ such that t is diagonalizable find a basis $\mathcal{B}_t$ of V_3 such that the matrix $m_{\mathcal{B}_t}^{\mathcal{B}_t}(T_{A_t})$ is diagonal

Solution. (a) The eigenvalues independent on t: they are $\lambda_1 = -5$ (double) and $\lambda_2 = 3$. The eigenspace $E(3)$ is one-dimensional, for each t. The eigenspace $E(-5)$ is the space of solutions of the homogenous system associated to the matrix $\begin{pmatrix} 0 & -t+1 & -1 \\ 0 & -8 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. This is two dimensional if and only if $-t+1 = -8$, that is $t = 9$. Therefore, we can find three independent eigenvectors if and only if $t = 9$. In conclusion, the matrix A_t is diagonalizable if and only if $t = 9$.
(b) For $t = 9$: $E(-5)$ is the space of solutions of the equation $-8y - z = 0$, that is $L((0,1,-8),(1,1,-8)$, while $E(3)$ is the space of solutions of the homogeneous system associated to the matrix $3I_3 - A_9 = \begin{pmatrix} 8 & -8 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix}$, that is $E(3) = L((1,1,0))$. In conclusion, $\mathcal{B}_9 = \{(0,1,-8),(1,1,-8),(1,-1,0)\}$. The matrix respesenting T_{A_9} with respect to this basis is: $diag(-5,-5,3)$.

2. Find an example of a non-triangular 3×3 matrix A whose characteristic polynomial is as follows: $P_A(\lambda) = (\lambda - 1)(\lambda + 2)(\lambda + 3)$.

Solution. Take a basis of V_3 (different from the standard basis), say $\mathcal{B}$ and take as $T_A : V_3 \to V_3$ the transformation such that $m_{\mathcal{B}}^{\mathcal{B}}(T_A) = diag(1,-2,-3)$. Let $C = m_{\mathcal{E}}^{\mathcal{B}}(id)$ e the matrix whose columns are the three vectors of $\mathcal{B}$. Then $diag(1,-2,-3) = C^{-1}AC$. Therefore

$$A = C\, diag(1,-2,-3)\, C^{-1}$$

Choosing a specific C (that is, a specific $\mathcal{B}$) you'll find your example. (Recall that similar matrices, that is matrices A, B such that $B = C^{-1}AC$ for some C, have the same eigenvalues and the same characteristic polynomial.)