L. A. G. test n. 6, june 9, 2016 Name:

1. Let $T : V_3 \to V_4$ defined by $T((1,1,0)) = (1,0,-1,0)$, $T((0,1,1)) = (0,1,0,1)$, $T((1,0,1)) = (3,-2,-3,-2)$. Compute dimension and a basis of $T(V_3)$ and dimension and a basis of $N(T)$.

Solution. It is easy to see that $\mathcal{B} = \{(1,1,0),(0,1,1),(1,0,1)\}$ is a basis of V_3. Therefore a linear transformation as defined in the text of the exercise exists and it is unique. Moreover, we know that $T(V_3) = L((1,0,-1,0),(0,1,0,1),(3,-2,-3,-2))$. We compute the dimension of this linear space by computing the rank of the matrix

$$m^{\mathcal{B}}_{\mathcal{E}_3}(T) : \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ -1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix}$$

which is clearly two. Therefore $\mathrm{rk}(T) = \dim T(V_3) = 2$ and a basis is given, for example, by the first two vectors, namely $\{(1,0,-1,0),(0,1,0,1)\}$.

Concerning $N(T)$, it has dimension 1 by the nullity + rank theorem. To find a basis of $N(T)$, we solve the homogeneous system associated to the matrix $m^{\mathcal{B}}_{\mathcal{E}_3}(T)$, and one find easily that the sapce of solutions is $L((-3,2,1))$. This means that

$$N(T) = L(-3(1,1,0) + 2(0,1,1) + (1,0,1)) = L((-2,-1,3)).$$

2. Compute the inverse of the matrix $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$

Solution.

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 2 & 0 & 0 & 0 & 1 & 1 & -1/3 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 1/2 & -1/6 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

Therefore the inverse is

$$\begin{pmatrix} 0 & 0 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & -1/6 \\ 0 & -1 & 0 & 1/3 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

3. Let V the linear space of all real polynomial of degree ≤ 2. Let us consider the following bases of V: $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x+1, (x+1)^2\}$.
Let $T : V \to V$ be the linear transformation defined by: $T(1) = x$, $T((x+1)) = x^2 + x$, $T((x+1)^2) = x + 2$.
(a) Write down the representing matrices $m^{\mathcal{C}}_{\mathcal{B}}(T)$, $m^{\mathcal{C}}_{\mathcal{C}}(T)$, $m^{\mathcal{B}}_{\mathcal{C}}(T)$, $m^{\mathcal{B}}_{\mathcal{B}}(T)$.
(b) Write down $T(1 - 3x + 2x^2)$.

Solution. By definition $m^{\mathcal{C}}_{\mathcal{B}}(T) = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

We can write $T(1) = x = -1 + (x+1)$, $T((x+1)) = x^2 + x = (x+1)^2 - (x+1)$, $T((x+1)^2) = x + 2 = (x+1) + 1$. Therefore

$$m^{\mathcal{C}}_{\mathcal{C}}(T) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

We can write $x = (x+1) - 1$ and $x^2 = (x+1)^2 - 2x - 1 = (x+1)^2 - 2(x+1) + 1$. Therefore, by linearity,

$$T(x) = T(x+1) - T(1) = (x+1)^2 - (x+1) - (-1 + (x+1)) = 1 - 2(x+1) + (x+1)^2,$$

and

$$T(x^2) = T((x+1)^2) - 2T(x+1) + T(1) = (x+1) + 1 - 2((x+1)^2 - (x+1)) + (-1 + (x+1)) =$$

$$= -2(x+1)^2 + 4(x+1)$$

Hence

$$m^{\mathcal{B}}_{\mathcal{C}}(T) = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

Finally, using the previous expressions, we get

$$T(x) = x^2 \qquad \text{and} \qquad T(x^2) = -2x^2 + 2$$

Therefore

$$m_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

(b) It is sufficient to multiply the matrix $m_{\mathcal{B}}^{\mathcal{B}}(T)$ and the vector $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$:

$$\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -7 \end{pmatrix}$$

Therefore $T(1 - 3x + 2x^2) = 4 + x - 7x^2$