L. A. G. **test n. 5,** **may 19, 2016** **Name:**

1. Let V be the linear space of all real polynomials of degree ≤ 2.
(a) Let $U = \{f(x) \in V \mid f'(1) + f(0) = 0\}$. Is U a linear subspace of V? If the answer is yes compute the dimension of U and find a basis of U.
(b) If U is a linear subspace, find another basis of U, different from the one found in (a).
(c) Let $Z = \{f(x) \in V \mid (2x+1)f'(x) = f(2x)\}$. Is Z a linear subspace of V? If the answer is yes compute the dimension of Z and find a basis of Z.

Solution. (a) Yes, U is a linear subspace:
(i) let $f, g \in U$. Then $(f+g)'(1) + (f+g)(0) = f'(1) + f(0) + g'(1) + g(0) = 0 + 0 = 0$ that is $f + g \in U$.
(ii) let $\lambda \in \mathbf{R}$ and $f \in U$. Then $(\lambda f)'(0) + (\lambda f)(0) = \lambda(f'(1) + f(0)) = 0$ that is $\lambda f \in U$.

Let $f(x) = a + bx + cx^2 \in V$. Then $f'(1) = b + 2c$ and $f(0) = a$. Therefore $f \in U$ if and only if $a + b + 2c = 0$, that is $a = -b - 2c$. Therefore

$$U = \{-b - 2c + bx + cx^2 \mid b, c \in \mathbf{R}\}.$$

Letting $b = 1$ and $c = 0$ we get $-1 + x$. Letting $b = 0$ and $c = 1$ we get $-2 + x^2$. Therefore $\dim U = 2$ and $\{-1 + x, -2 + x^2\}$ is a basis of U .
(b) One can take two linearly independent linear combinations of the elements of the previous basis, for example $\{-3 + x + x^2, 1 + x - x^2\}$.
(c) Yes, Z is a linear subspace:
(i) let $f, g \in Z$. Then

$$(2x+1)(f+g)'(x) = (2x+1)f'(x) + (2x+1)g'(x) = f(2x) + g(2x) = (f+g)(2x).$$

Therefore $f + g \in Z$.
(ii) Let $\lambda \in \mathbf{R}$ and $f \in Z$. Then $(2x+1)(\lambda f)'(x) = \lambda(2x+1)f'(x) = (\lambda f)(2x)$. Therefore $\lambda f \in Z$.

Let $f(x) = a + bx + cx^2 \in V$. Then $(2x+1)f'(x) = (2x+1)(b+2cx) = b + 2(b+c)x + 4cx^2$. Moreover $f(2x) = a + 2bx + 4cx^2$. Therefore $f(x) \in Z$ if and only of

$$\begin{cases} a & = \text{b} \\ 2b + 2c & = \text{2b} \\ 4c & = \text{4c} \end{cases}$$

Therefore $c = 0$ and $a = b$. Hence $Z = \{a + ax \mid a \in \mathbf{R}\} = L(1 + x)$. Hence $\dim Z = 1$ and $\{1 + x\}$ is a basis.

2. Let V be the space of all real polynomials of degree ≤ 2, equipped with inner product $(f, g) = \int_{-1}^{1} f(x)g(x)dx$, and let $W = \{f(x) \in V \mid f(1) = 0\}$. Find the polynomial in W which is closest to $g(x) = x$ and compute its distance from $g(x)$.

Solution. Bases of W is easily found: for example: $\{(x-1,(x-1)^2\}$: it is enough to find two linearly independent polynomials having $x=1$ as zero. Other examples: $\mathcal{B}_a = \{(x-1),(x-1)(x-a)\}$, for any $a \in \mathbf{R}$. Otherwise, you can compute a basis as in the previous exercise. Let us choose the basis $\{x-1, x^2-1\}$, which might be easy for calculations (however there are other good choices).

Probably the fastest way to answer consists in computing $W^\perp$ first (note that $\dim W^\perp = 1$). Easy and very fast calculations show that:

$$(x^2, x-1) = -\frac{2}{3}, \quad (x, x-1) = \frac{2}{3} \quad (1, x-1) = -2 \quad (x^2, x^2-1) = -\frac{4}{15}$$

$$(x, x^2-1) = 0 \quad (1, x^2-1) = -\frac{4}{3}$$

A polynomial $f(x) = a + bx + cx^2$ belongs to $W^\perp$ if and only if

$$\begin{cases} (f(x), x-1) = 0 \\ (f(x), x^2-1) = 0 \end{cases}$$

that is, using the above calculations

$$\begin{cases} -2a + \frac{2}{3}b - \frac{2}{3}c = 0 \\ -\frac{4}{3}a - \frac{4}{15}c = 0 \end{cases}$$

Solving the system one gets $b = -8a$ and $c = -5a$. Therefore

$$W^\perp = L(1 - 2x - 5x^2)$$

Therefore

$$p_{W^\perp}(x) = \frac{(x, 1-2x-5x^2)}{(1-2x-5x^2),(1-2x-5x^2)}(1-8x-5x^2) = \frac{\frac{-4}{3}}{8}(1-2x-5x^2) = -\frac{1}{6}(1-2x-5x^2)$$

Therefore the distance between x and W is

$$\| -\frac{1}{6}(1-2x-5x^2) \| = \frac{1}{6}\sqrt{8} = \frac{\sqrt{2}}{3}$$

and the closest point is

$$h(x) = x - \left(-\frac{1}{6}(1-2x-5x^2)\right) = \frac{1}{6} + \frac{2}{3}x - \frac{5}{6}x^2 = \frac{1}{6}(1+4x-5x^2)$$