L. A. G. **test n. 4,** **may 6, 2016** **Name:**

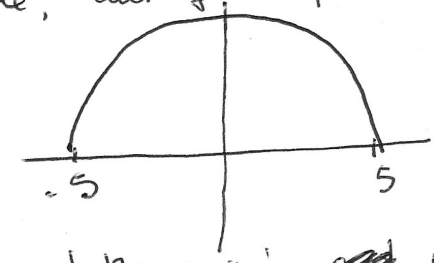
1. (a) Let $f : I \to \mathbf{R}$ be a real-valued function, defined on an interval $I \subseteq \mathbf{R}$. Find a formula for the curvature and the unit normal vector of the graph of f (pay attention to the direction of the unit normal vector).
(b) Find an example of a function $f : [-5, 5] \to \mathbf{R}$ such that: $f(-5) = f(5) = 0$ and the graph of f has constant positive curvature.

2. A point moves in V_3 in such a way that $\| \mathbf{v}(t) \|$ is constant, equal to 5, and $\| \mathbf{a}(t) \|$ is constant, equal to 2. Compute the curvature function $\kappa(t)$.

SOLUTION

Ex. 2/ Since $\|\underline{v}(t)\|$ is constant then $\|\underline{a}(t)\|$ is always orthogonal to $\underline{v}(t)$. Hence $\|\underline{a}(t) \times \underline{v}(t)\| = \|\underline{a}(t)\| \, \|\underline{v}(t)\|$.
Therefore $\kappa(t) = \dfrac{\|a(t)\| \, \|v(t)\|}{\|v(t)\|^3} = \dfrac{2}{25}$.

Ex. 1/ (b) We know that if a plane curve has constant positive curvature it must be a portion of a circle. Therefore we have to find an example of a function $f : [-5, 5]$ such that its graph is a portion of a circle, and $f(-5) = f(5) = 0$. The simplest example is the half-circle of radius 5.



Since the circle ~~is~~ centered at the origin ~~and~~ with radius $= 5$ has equation $x^2 + y^2 = 25$, we have $y = \pm\sqrt{25 - x^2}$.
Hence $f(x) = \sqrt{25 - x^2}$ is a function ~~accepted~~ as required.

(a) The parametrization is $\underline{r}(x) = (x, f(x))$. Therefore (as we know) $\underline{v}(x) = (1, f'(x))$ and $\underline{a}(x) = (0, f''(x))$.

$$v(x) = \sqrt{1 + f'(x)^2}$$

Therefore the curvature at the point $(x, f(x))$ is:

$$K(x) = \frac{\|\underline{a}(x) \times \underline{v}(x)\|}{v^3(x)} = \frac{|f'(x) f''(x)|}{(1 + f'(x)^2)^{3/2}}$$

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SOLUTION (CONTINUATION)

The unit normal vector is perpendicular to $\underline{v}(x) = (1, f'(x))$ hence

$$N(x) = \pm \frac{(-f'(x), 1)}{\sqrt{1+(f'(x))^2}}$$

The only point is to decide when it's "+" and when it's "−".
The final answer is:
$$N(x) = \begin{cases} \dfrac{(-f'(x), 1)}{\sqrt{1+(f'(x))^2}} & \text{if } f''(x) > 0 \\ -\dfrac{(-f'(x), 1)}{\sqrt{1+(f'(x))^2}} & \text{if } f''(x) < 0 \end{cases}$$

You can arrive to this answer in various ways:

Way 1) With calculations: project $\underline{a}(x)$ along $\underline{v}(x)$:

$$(*) \quad (0, f'') = \frac{f'f''}{(1+f'^2)}(1, f') + D \quad \text{with } D \cdot \underline{v} = 0$$

we know that $N = \dfrac{D}{\|D\|}$ (we have seen this: just compare $(*)$ with the formula:

$$\underline{a} = v' T + v \|T'\| N$$)

From $(*)$ we get $D = \dfrac{1}{1+f'^2}(-f'f'', f'') = \dfrac{f''}{1+f'^2}(-f', 1)$

Therefore $\|D\| = \left|\dfrac{f''}{1+f'^2}\right| \sqrt{1+f'^2} = \begin{cases} \dfrac{f''}{\sqrt{1+f'^2}} & \text{if } f'' > 0 \\ -\dfrac{f''}{\sqrt{1+f'^2}} & \text{if } f'' < 0 \end{cases}$

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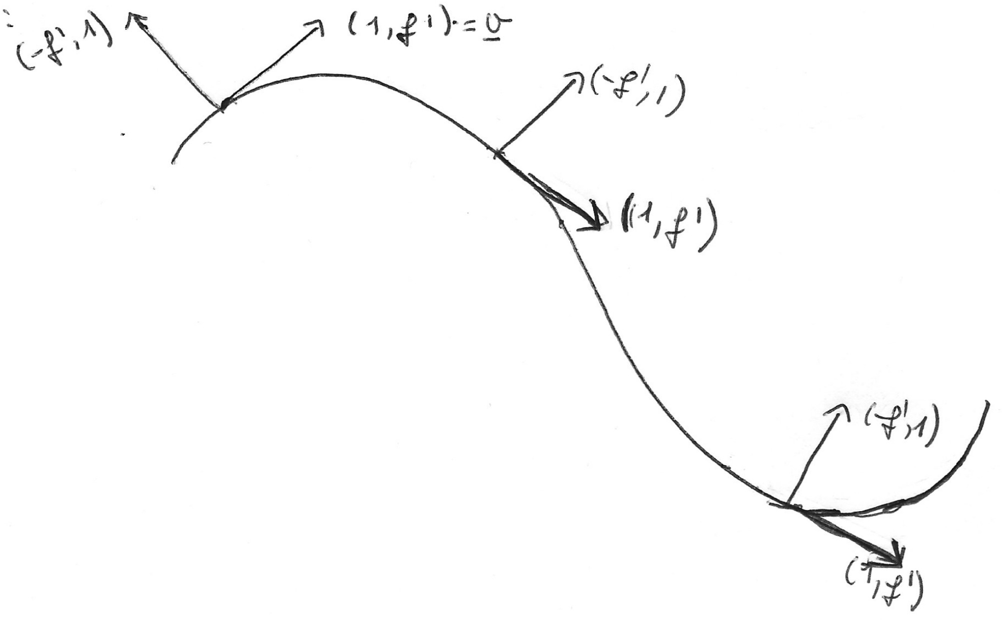
<u>SOLUTION</u> (CONTINUATION)

In conclusion
$$N = \frac{D}{\|D\|} = \begin{cases} \dfrac{(-f', 1)}{\sqrt{1+f'^2}} & \text{if } f'' > 0 \\[2ex] -\dfrac{(-f', 1)}{\sqrt{1+f'^2}} & \text{if } f'' < 0. \end{cases}$$
$\square$.

<u>2° WAY</u>]. The orthogonal basis $\{(1, f'(x)), (-f'(x), 1)\}$ is "oriented as the basis given by the unit vectors of the coordinate axes", that is the second vector is obtained from the first one with a ~~..~~ counterclockwise rotation of 90°:



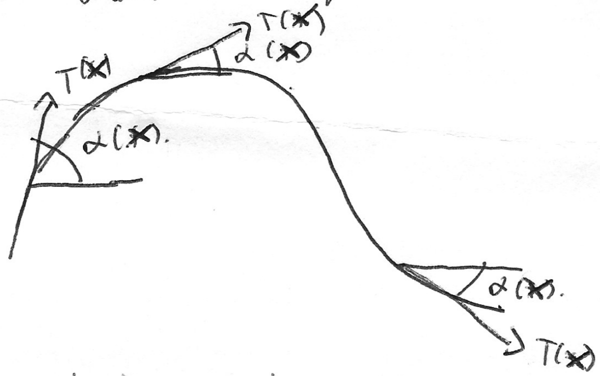
SOLUTION (continuation)

But we know (end of § 14.8) that the unit normal vector always points toward the interior of the curve.

Therefore $N(x) = \begin{cases} \dfrac{(-f'(x),1)}{\sqrt{1+(f'(x))^2}} & \text{if the graph is convex at } (x,f(x)) \\ -\dfrac{(-f'(x),1)}{\sqrt{1+(f'(x))^2}} & \text{if the graph is concave} \end{cases}$

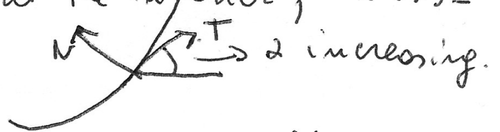
But we know from MA1 that the graph is convex when $f''(x)>0$ and concave when $f''(x)<0$. In this way we get the final answer.

3° way ~~Bet~~ As at the end of § 14.8, let us denote $\alpha(x)$ the angle between $T(x)$ and the x-axis.



Since $\underline{v}(x) = (1, f'(x))$ we have that $f'(x)$ is $\operatorname{tg}\alpha(x) = \dfrac{\sin\alpha(x)}{\cos\alpha(x)}$

~~Since~~ Therefore if $f''(x)>0$ then $f'(x)$ is increasing, hence $\alpha(x)$ is also increasing. Therefore, since $N(x)$ points toward the interior, $N(x) = \dfrac{(-f'(x),1)}{\sqrt{1+(f'(x))^2}}$!



If $f''(x)<0$ then $f'(x)$ is decreasing therefore $\alpha(x)$ is decreasing, too. Hence $N(x) = -\dfrac{(-f'(x),1)}{\sqrt{1+(f'(x))^2}}$