L. A. G. test n. 2, april 7, 2016 Name:

1. Let P, Q, B be elements of V_3, and suppose that $B \neq O$. Let us consider the line $R = \{P + tB \mid t \in \mathbf{R}\}$ and denote $S_{Q,R}$ the point of R which is nearest to Q.

(a) Find a formula expressing $S_{Q,R}$ in function of Q, P and B.

(b) Apply your formula to find $S_{Q,R}$ when $Q = (1, -2)$ and $R = \{(2, 2) + t(1, -1) \mid t \in \mathbf{R}\}$. Compute also the distance between Q and the line R in this case.

Solution. (a) the nearest point is P plus the projection of $(Q - P)$ on B. Hence:

$$S_{Q,R} = P + (\frac{(Q - P) \cdot B}{B \cdot B})B$$

(b)

$$S_{Q,R} = (2, 2) + (\frac{(-1, -4) \cdot (1, -1)}{2})(1, -1) = \frac{1}{2}(7, 1)$$

Hence

$$d(Q, R) = \| Q - S_{Q,R} \| = \| (1, -2) - \frac{1}{2}(7, 1) \| = \| \frac{1}{2}(-5, -5) \| = \frac{5\sqrt{2}}{2}$$

2. Let $A = (1, 2, 0)$, $B = (0, 1, 1)$, $D = (1, 3, 3)$ and, for t varying in $\mathbf{R}$, $C_t = (1, t, 2t)$. For which values of t the vector D belongs to $L(A, B, C_t)$?

Solution. If A, B and C_t are linearly independent then $L(A, B, C_t) = V_3$. Therefore *all* vectors of V_3, including D, belong to $L(A, B, C_t)$. We know that A, B and C_t are linearly independent if and only if $\det \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & t & 2t \end{pmatrix} \neq 0$. Computing the determinant we find $t+2$. Therefore $D \in L(A, B, C_t)$ when $t \neq -2$.

The case $t = -2$ needs to be examined separately. To simplify the calculations, one can argue as follows: in this case the three vectors are dependent. Since A and B are independent, C_{-2} is linear combination of A and B. Hence $L(A, B, C_{-2}) = L(A, B)$. Therefore it is sufficient to verify whether D belongs to $L(A, B)$ or not. That is, we have to verify whether the vector equation

$$xA + yB = D$$

has a solution. This is the system

$$\begin{cases} x & = 1 \\ 2x + y & = 3 \\ y & = 3 \end{cases}$$

which clearly does not have a solution.

Final answer: D belongs to $L(A, B, C_t)$ if and only if $t \neq -2$.