L. A. G.   test n. 1,   march 16, 2016   Name:

1. (a) Let $A = (-1, 3, 2)$ and $B = (2, 2, -1)$. Find $C \in L(B)$ such that $(0, 0, 0)$, A and C are the vertices of a right triangle.
(b) Compute the cosine of the angle between A and B.
(c) Find all unit vectors parallel to A.

SOLUTION. (a) A and B are not orthogonal. Since C must be cB, for some $c \in \mathbf{R}$, also A and C can't be orthogonal. Therefore, the vertex corresponding to the right angle can't be O. hence it will be either A or C. This leads to two different solutions.

The easiest way is to look for a $C \in L(B)$ such that O, A and C are the vertices of a right triangle *and the vertex corresponding to the right angle is* C. For this one takes as C the projection of A along B:

$$C = (\frac{A \cdot B}{B \cdot B})B = \frac{2}{9}(2, 2, -1).$$

Then, with this choice of C:

$$C \in L(B) \qquad \text{and} \qquad (A - C) \cdot C = 0$$

which means exactly that the triangle with vertices O, A and C has a right angle at the vertex C.

(b)

$$\cos\theta = \frac{A \cdot B}{\| A \| \| B \|} = \frac{2}{\sqrt{14}\sqrt{9}}$$

(c) $\| cA \| = |c| \ \| A \|$. Therefore $\| cA \| = 1$ if and only if $|c| = \frac{1}{\|A\|}$. Hence the vectors as requested are two:

$$\frac{1}{\sqrt{14}}A \qquad \text{and} \qquad -\frac{1}{\sqrt{14}}A$$

2. For the following choices of subsets $S \subset V_3$ say which of the statements below is correct, and explain why.
(a) $S = \{(1, 1, 0), (1, 2, 1), (0, 1, 1)\}$.
(b) $S = \{(1, 1, 0), (1, 2, 1)\}$.
(c) $S = \{(1, 1, 0), (1, 2, 1), (1, 0, 3)\}$.

∘ Every vector in V_3 is spanned uniquely by S. (Note: spanned=generated).
∘ No vector in V_3 is spanned uniquely by S.

∘ There are some vectors in V_3, but not all, which are spanned uniquely by S.

SOLUTION. We need to understand whether S is an independent subset or a dependent subset.
(a) Is S independent or dependent? This means: does the equation

$$x(1,1,0)+y(1,2,1)+z(0,1,1)=(0,0,0)$$

has only $x=y=z=0$ as solution, or there are more solutions? Solving the corresponding system with gaussian elimination we get:

$$\begin{pmatrix}1&1&0\\1&2&1\\0&1&1\end{pmatrix}\to\begin{pmatrix}1&1&0\\0&1&1\\0&1&1\end{pmatrix}$$

This means that the system is equivalent to $\begin{cases}x+y&=0\\y+z&=0\end{cases}$ which has clearly infinitely many solutions. For example: $(1,-1,1)$, that is

$$(1,1,0)-(1,2,1)+(0,1,1)=(0,0,0)$$

Therefore the subset S is *dependent.*
The correct statement is: *No vector in V_3 is spanned uniquely by S.*
The reason is Theorem 12.7 of Section 12.12 of the textbook: either $D\notin L(S)$ or, if $D\in L(S)$, it is not spanned uniquely.
Concretely: suppose that $D=a(1,1,0)+b(1,2,1)+c(0,1,1)$.
Since $(1,1,0)-(1,2,1)+(0,1,1)=(0,0,0)$, D can be written also as:

$$D=(a+1)(1,1,0)+(b-1)(1,2,1)+(c+1)(0,1,1).$$

(b) Is S independent or dependent? Since the two vectors are clearly non-parallel, it is independent.
Therefore the correct statement is: *There are some vectors in V_3, but not all, which are spanned uniquely by S.*
Indeed, by Thm 12.7 every vector of $L(S)$ is spanned uniquely. However not every vector belongs to $L(S)$. Otherwise S would be a basis of V_3, which is impossible since it has only two elements (see Theorem 12.10 of Section 12.14).

(c) Is S independent or dependent? As in (a):

$$\begin{pmatrix}1&1&1\\1&2&0\\0&1&3\end{pmatrix}\to\begin{pmatrix}1&1&1\\0&1&-1\\0&1&3\end{pmatrix}\to\begin{pmatrix}1&1&1\\0&1&-1\\0&0&2\end{pmatrix}$$

Therefore,as it is easily checked, $x=y=z=0$ is the only solution. Hence S is *independent.*
Being this the case, the correct statement is: *Every vector in V_3 is spanned uniquely by S.*
Indeed S is a basis of V_3.