L. A. G. year 2016-'17 final exam n.8 , march 27, 2017 Name:

1. Let L be the line defined by $\begin{cases} x+y+5z & =\text{-}2 \\ x-y-z & =0 \end{cases}$.
Ler R be the line $R = \{(1,2,-1)+t(-2,1,1)\}$. Moreover let $P=(1,1,1)$.
(a) Is there a plane containing both L and P? If the answer is yes, find its cartesian equation.
(b) Compute $L \cap R$.
(c) Is there a plane containing both L and R? If yes, find its cartesian equation.

Solution. (a) The answer is YES in any case (if P is not contained in L there is a unique plane containing L and P. If P belongs to L, infinitely many planes). We have:

$$\begin{cases} x+y+5z & =\text{-}2 \\ x-y-z & =0 \end{cases} \to \begin{cases} x+y+5z & =\text{-}2 \\ -y-3z & =0 \end{cases}$$

Therefore $y=-3z$ and $x=-2+3z-5z$. In conclusion, the line L: $(-2-2z,-3z,z) = (-2,0,0)+z(-2,-3,1) = Q+zA$. The required plane is

$$Q+t(P-Q)+sA = (-2,0,0)+t(3,1,1)+z(2,-3,1), \qquad t,s \in \mathbf{R}$$

Its cartesian equation is

$$4x-y-11z=-8$$

(b) Substituting the parametric equation of R into the cartesian equation of L we get $t=0$. Therefore $L \cap R = (1,2-1)+0(-2,1,1) = (1,2,-1)$.
(c) The answer is YES. The parametric equation of the required plane is
$(1,2,-1)+t(-2,-3,1)+s(-2,1,1)$. The cartesian equation: $x+2z=-1$.

2. Let us consider a parabola $\mathcal{C}$ with focus in the origin, directrix of equation $4x-3y=k$ and such that $\mathcal{C}$ contains the point $(1,0)$.
(a) Assume that $k>0$. Find k. Find the vertex of $\mathcal{C}$.
(b) Assume that $k<0$. Find k.

Solution. Second session 2012.

3. Let W be the linear subspace of V_4 defined by the equation $x+y-z+t=0$. Find an orthogonal basis of V_4 such that three of its vectors belong to W.

Solution. The vector $\mathbf{u}=(1,1,-1,1)$ is orthogonal to W. We can take $\mathbf{u}$ as one of the vectors of the required basis. To construct the other three vectors as requested we need to find a basis of W and orthogonalize it. To find a basis of W: setting $y=1$, $z=t=0$

we get $(1,-1,0,0)$. Setting $y=t=0$ and $z=1$ we get $(1,0,1,0)$. Setting $y=z=0$ and $t=1$ we get $(1,0,0,-1)$. Hence a basis of W is $\{(1,-1,0,0),(1,0,1,0),(1,0,0,-1)\}$.
Now we orthogonalize it:
we take $\mathbf{w}_1=(1,-1,0,0)$
$(1,0,1,0)-\frac{1}{2}(1,-1,0,0)=(\frac{1}{2},\frac{1}{2},1,0)$. We take $bfw_2=(1,1,2,0)$.
$(1,0,0,-1)-\frac{1}{2}(1,-1,0,0)-\frac{1}{6}(1,1,2,0)=(\frac{1}{3},\frac{1}{3},-\frac{1}{3},-1)$.
In conclusion, as basis as required is

$$\{(1,-1,0,0),(1,1,2,0),(1,1,-1,-3),(1,1,-1,1)\}$$

4. Let us consider the following basis of V_3: $\mathcal{B}=\{\mathbf{u},\mathbf{v},\mathbf{w}\}=\{(1,1,0),(1,0,1),(1,0,-1)\}$. Let $T:V_3\to V_3$ be the linear transformation such that $T(\mathbf{u})=(1,0,0)$, $T(\mathbf{v})=(0,1,0)$, $T(\mathbf{w})=(0,0,0)$. Let $\mathcal{E}=\{(1,0,0),(0,1,0),(0,0,1)\}$ be the standard basis of V_3.
(a) Find the representative matrices $m^{\mathcal{B}}_{\mathcal{E}}(T)$, and $m^{\mathcal{E}}_{\mathcal{E}}(T)$.
(b) Compute $T((1,0,0))$.
(c) Compute the null-space of T.
(d) Is T one-to-one? If the answer is yes explain why. If the answer is no, find explicitly two vectors $\mathbf{u}_1\neq\mathbf{u}_2$ of V_3 such that $T(\mathbf{u}_1)=T(\mathbf{u}_2)$.

Solution. (c) Clearly $N(T)=L(\mathbf{w})=L((1,0,-1))$.
(a) and (b) Clearly $m^{\mathcal{B}}_{\mathcal{E}}(T)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$.
The matrix $m^{\mathcal{E}}_{\mathcal{E}}(T)$ can be found using the change-of-basis formula. However in this case the calculations are so easy that it can be found directly. In fact one sees immediately that $(1,0,0)=\frac{1}{2}\mathbf{v}+\frac{1}{2}\mathbf{w}$. Therefore $T((1,0,0))=\frac{1}{2}T(\mathbf{v})=(0,\frac{1}{2},0)$. (This answers to question (c)).
$(0,1,0)=\mathbf{u}-(\frac{1}{2}\mathbf{v}+\frac{1}{2}\mathbf{w})$. Hence $T((0,1,0))=T(\mathbf{u})-\frac{1}{2}T(\mathbf{v})=(1,-\frac{1}{2},0)$.
Finally: $(0,0,1)=\frac{1}{2}\mathbf{v}-\frac{1}{2}\mathbf{w}$. Therefore $T((0,0,1))=\frac{1}{2}T(\mathbf{v}=(0,\frac{1}{2},0)$.
In conclusion $m^{\mathcal{E}}_{\mathcal{E}}(T)=\begin{pmatrix}0&1&0\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&0&0\end{pmatrix}$.

5. Let $\mathbf{u}=(1,1,0)$, $\mathbf{v}=(1,0,1)$, $\mathbf{w}=(1,0,-1)$. Let $T:V_3\to V_3$ be the linear transformation defined by: $T(\mathbf{u})=3\mathbf{u}$, $T(\mathbf{v})=-4\mathbf{v}$, $T(\mathbf{w})=\mathbf{v}+\mathbf{w}$. Find eigenvalues and eigenvectors of T. Is T diagonalizable? If the answer is no, explain why. If the answer is yes, find a basis $\mathcal{B}$ of V_3 such that the matrix representing T with respecting to the basis $\mathcal{B}$ is diagonal.

Solution. Let us denote $\mathcal{C}$ the basis $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$. The matrix representing T with respect to $\mathcal{C}$ is

$$M=m^{\mathcal{C}}_{\mathcal{C}}(T)=\begin{pmatrix}3&0&0\\0&-4&1\\0&0&1\end{pmatrix}$$

Therefore we get that:
$\lambda_1 = 3$ is an eigenvalue, of eigenspace $E(3) = L(\mathbf{u}) = L((1,1,0))$ (this was already clear from the beginning);
$\lambda_2 = -4$ is an eigenvalue, of eigenspace $E(-4) = L(\mathbf{v}) = L((1,0,1))$ (this was already clear from the beginning);
The other eigenvalue is $\lambda_3 = 1$ (this can be seen by computing the characteristic polynomial of $m_{\mathcal{C}}^{\mathcal{C}}(T)$, or simply the trace of $m_{\mathcal{C}}^{\mathcal{C}}(T)$). Since the matrix $1I_3 - M = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ we get, solving the corresponding homogeneous linear system, that $E(1) = L(0\mathbf{u} + \mathbf{v} + 5\mathbf{w}) = L(\mathbf{v} + 5\mathbf{w}) = L((1,0,-4))$.
The linear transformation T is diagonalizable. Letting $\mathcal{B}$ the basis $\{\mathbf{u}, \mathbf{v}, \mathbf{v} + 5\mathbf{w}\}$ we have that $m_{\mathcal{B}}^{\mathcal{B}}(T) = diag(4, -3, 1)$.