L. A. G. final exam n.4, sept 30, 2016 Name:

1. In V_2 let us consider the following straight lines

$$L = \{(1,2) + t(1,1) \mid t \in \mathbf{R}\} \qquad \text{and} \qquad R = \{(1,2) + s(1,3) \mid s \in \mathbf{R}\}.$$

Find the cartesian equations of all lines S parallel to the vector $(-2,1)$ such that the area of the triangle formed by the lines L, R and S is equal to 10.

Solution. The cartesian equation of a line S_c parallel to the vector $(-1,2)$ is of the form $2x + y = c$. Substituting $x = 1 + t$ and $y = 2 + t$ one gets $t = (c-5)/3$. Therefore the intersection with L is $(1,2) + ((c-5)/3)(1,1)$. Similarly, the intersection of S_c with R is $(1,2) + ((c-5)/7)(1,3)$.

The vertices of the triangle are $(1,2)$, $(1,2)+((c-5)/3)(1,1)$, $(1,2)+((c-5)/7)(1,3)$. The area of the triangle is

$$\frac{1}{2} \parallel \frac{c-5}{3}(1,1,0) \times \frac{c-5}{7}(1,3,0) \parallel = \frac{1}{21}|(c-5)^2| \parallel (0,0,2) \parallel = 2\frac{(c-5)^2}{21}$$

Therefore we are looking for the solutions of the equation

$$(c-5)^2 = 105$$

That is $c = 5 + -\sqrt{105}$.

2. Let $V = L((1,0,1,1),(0,1,1,1))$ and $\mathbf{w} = (1,2,-1,-2)$. Write $\mathbf{w}$ as the sum of a vector in V and a vector perpendicular to V (with respect to the usual dot product of V_4).

Solution. To answer the question is tantamount to find the orthogonal decomposition of $\mathbf{w}$ with respect to V.

We first find an orthogonal basis if V with the method of Gram-Schmidt. One can take, for example: $\{1,0,1,1),(-2,3,1,1)\}$. The orthogonal projection of $\mathbf{w}$ on V is the sum of the ortogonal projections of $\mathbf{w}$ on the two vectors of the orthogonal basis:

$$p_V(\mathbf{w}) = \frac{-2}{3}(1,0,1,1) + \frac{1}{15}(-2,3,1,1) = \frac{1}{15}(-12,3,-9,-9).$$

the required decomposition is

$$\mathbf{v} = p_V(\mathbf{v}) + (\mathbf{v} - p_V(\mathbf{v})) = \frac{1}{15}(-12,3,-9,-9) + \frac{1}{15}(27,27,-6,-21)$$

3. A particle moves in V_3 with position vector

$$\mathbf{r}(t) = (t\,,\, t^2\,,\, \frac{4}{3}t^{\frac{3}{2}})$$

Find how long it takes to cover an arc of length 12 starting from the initial position $\mathbf{r}(0)$.

Solution. $\mathbf{v}(t) = (1, 2t, 2t^{1/2})$. Therefore $v(t) = \sqrt{1 + 4t^2 + 4t} = |1 + 2t|$. Therefore we have to find the $t \geq 0$ such that $\int_0^t (1 + 2x)dx = 12$. Therefore $t + t^2 = 12$ whose positive solution is $\bar{t} = 3$.

4. Let us consider the following basis of V_4:

$$\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{r}\} = \{(1, 0, 1, 1)\,,\,(1, 1, 3, -1)\,,\,(1, -1, 1, 1)\,,\,(-1, 1, 1, 1)\}$$

Let $T : V_4 \to V_4$ be the linear transformation defined by $T(\mathbf{u}) = 3\mathbf{u}$, $T(\mathbf{v}) = 6\mathbf{u}$, $T(\mathbf{w}) = -5\mathbf{w}$, $T(\mathbf{r}) = 3\mathbf{u} - 5\mathbf{w}$.
(a) Compute null-space, eigenspaces and eigenvalues of T. Is T diagonalizable?
(b) Let $L : V_4 \to V_4$ be the linear transformation defined by $L(X) = 3T(X) - 5X$. Compute eigenvalues and eigenspaces of L. Is L invertible? Is L diagonalizable? Compute the trace and the determinant of the linear transformation L.

Solution. (a) One sees immediately that 3 and -5 are eigenvalues, of eigenvectors $\mathbf{u}$ and $\mathbf{w}$ respectively. Moreover $T(\mathbf{v}) = 2T(\mathbf{u})$, that is $T(\mathbf{v} - 2\mathbf{u}) = O$. Similarly, $T(\mathbf{r} - \mathbf{u} - \mathbf{w}) = O$. Therefore 0 is an eigenvalue (equivalently: there is a non-trivial null-space) and the eigenspace of 0, that is, the null-space of T, is $L(\mathbf{v} - 2\mathbf{u}, \mathbf{r} - \mathbf{u} - \mathbf{w})$. Therefore the eigenvalues are: 3,-5,0(double) and T is diagonalizable. A basis of eigenvectors is $\{\mathbf{u}, \mathbf{w}, \mathbf{v} - 2\mathbf{u}, \mathbf{r} - \mathbf{u} - \mathbf{w}\}$.
(b) The eigenvalues and eigenvectors of L are easily computed from those of T: $L(\mathbf{u}) = 3T(\mathbf{u}) - 5\mathbf{u} = 9\mathbf{u} - 5\mathbf{u} = 4\mathbf{u}$. Similarly $T(\mathbf{w}) = -20\mathbf{w}$. Moreover, if X belongs to the null-space of T, $L(X) = O - 5X = -5X$. Therefore the eigenvalues of L are: 4, -20 and -5 (double), and L is diagonalizable with the same basis as for T.

One finds the same results using the matrices $m^{\mathcal{B}}_{\mathcal{B}}(T) = \begin{pmatrix} 3 & 6 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $m^{\mathcal{B}}_{\mathcal{B}}(L) = \begin{pmatrix} 9-5 & 18 & 0 & 9 \\ 0 & 0-5 & 0 & 0 \\ 0 & 0 & -15-5 & -15 \\ 0 & 0 & 0 & -5 \end{pmatrix}$

L invertible? Yes: the determinant is 2000, the product of the eigenvalues 4, -20, -5 and -5, The trace is the sum: -26.

Trace

5. Let $\mathcal{C}$ be the conic of equation

$$2x^2 + 4xy + 2y^2 - x + 2y - \frac{31}{32} = 0\,.$$

Reduce the equation to canonical form, find the coordinates, with respect to the standard basis of V_2, of the center/vertex and find the equations/equation of the symmetry axes/axis. Draw a rough sketch of $\mathcal{C}$ in the reference system given by the standard basis.

Solution. The matrix of the quadratic part is $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Its eigenvalues are 4 and 0. Taking the eigenvalues in this order, an orthonormal basis of eigenvectors is $\mathcal{B} = \{(\frac{1}{\sqrt{2}}(1,1), (\frac{1}{\sqrt{2}}(-1,1)\}$. The change-of-basis matrix is $m_{\mathcal{E}}^{\mathcal{B}}(id) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Therefore the change of coordinates is

$$\begin{cases} x = \frac{1}{\sqrt{2}}(x' - y') \\ y = \frac{1}{\sqrt{2}}(x' + y') \end{cases}$$

Substituting in the equation of the conic we get

$$4{x'}^2 + \frac{1}{\sqrt{2}}x' + \frac{3}{\sqrt{2}}y' - \frac{31}{32}$$

Completing the square, we have that

$$4{x'}^2 + \frac{1}{\sqrt{2}}x' = 4({x'}^2 + \frac{1}{4\sqrt{2}}x') = 4((x' + \frac{1}{8\sqrt{2}})^2 - \frac{1}{128}) = 4(x' + \frac{1}{8\sqrt{2}})^2 - \frac{1}{32}$$

Therefore we get

$$4(x' + \frac{1}{8\sqrt{2}})^2 - \frac{1}{32} + \frac{3}{\sqrt{2}}y' - \frac{31}{32} = 4(x' + \frac{1}{8\sqrt{2}})^2 + \frac{3}{\sqrt{2}}y' - 1 = 4(x' + \frac{1}{8\sqrt{2}})^2 + \frac{3}{\sqrt{2}}(y' - \frac{\sqrt{2}}{3})$$

The conic is a parabola of canonical equation

$$(y' - \frac{\sqrt{2}}{3}) = \frac{4\sqrt{2}}{3}(x' + \frac{1}{8\sqrt{2}})^2$$

The (x', y') coordinates of the vertex are $(\frac{\sqrt{2}}{3}, -\frac{1}{8\sqrt{2}})$. One finds the (x, y) coordinates usingthe change of coordinates.

The symmetry axis is the line passing trough the vertex, parallel to the vector $(1, -1)$.