L. A. G. final exam n.2, july 13, 2016 Name:

1. Let us consider the plane $M = \{(1,0,-1) + s(1,2,-1) + t(0,3,-1) \mid s,t \in \mathbf{R}\}$.
For a, b varying in $\mathbf{R}$ let us consider the line $L_{a,b} = \{\, (1, a+1, 2) + \lambda(b, 1, -2) \mid \lambda \in \mathbf{R}\,\}$.
Find the values of a and b such that the line $L_{a,b}$ is contained in the plane M.

Solution. If $L_{a,b}$ is contained in M if and only if the vector $(b, 1, -2) \in L((1,2,-1),(0,3,-1))$, and the point $(1, a+1, 2)$ belongs to M.
Concerning the first condition, it means that there are $s, t \in \mathbf{R}$ such that

$$s(1,2,-1) + t(0,3,-1) = (b,1,-2).$$

The system corresponds to the augmented matrix

$$\begin{pmatrix} 1 & 0 & b \\ 2 & 3 & 1 \\ -1 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & b \\ 0 & 3 & 1-2b \\ 0 & -1 & -2+b \end{pmatrix} \to \begin{pmatrix} 1 & 0 & b \\ 0 & -1 & -2+b \\ 0 & 0 & 1-2b+3(-2+b) = -5+b \end{pmatrix}$$

From this we see that the system has solution if and only if $b = 5$.
Next we see for which values of a the point $(1, a+1, 2)$ belongs to the plane M. One finds $a = -10$.
Conclusion: the line $L_{a,b}$ is contained in M for the pair $(a,b) = (-10, 5)$.

2. Let $\mathbf{u} = (1,2,-1)$, $\mathbf{v} = (-3,2,2)$, $\mathbf{w} = (13,2,-10)$. Let $\mathbf{a} = 18\mathbf{u} - 19\mathbf{v} + 15\mathbf{w}$.
Write $\mathbf{a}$ as linear combination of $\mathbf{u}$ $\mathbf{v}$ and $\mathbf{w}$ in a different way. (Warning: you are NOT required to compute $\mathbf{a}$. You are required to write $\mathbf{a}$ as a different linear combination of $\mathbf{u}$ $\mathbf{v}$ and $\mathbf{w}$).

Solution. The relevant theory is Theorem 12.7, Vol. I, on linear independence.
The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a dependent set. To see this we consider the homogeneous linear system $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = O$. This corresponds to the matrix

$$\begin{pmatrix} 1 & -3 & 13 \\ 2 & 2 & 2 \\ -1 & 2 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 13 \\ 0 & 8 & -24 \\ 0 & -1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 13 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

The solutions are $(-4z, 3z, z)$, $z \in \mathbf{R}$. For example $-4\mathbf{u} + 3\mathbf{v} + \mathbf{w} = O$. Therefore

$$\mathbf{a} = 18\mathbf{u} - 19\mathbf{v} + 15\mathbf{w} = 18\mathbf{u} - 19\mathbf{v} + 15\mathbf{w} + O =$$

$$= 18\mathbf{u} - 19\mathbf{v} + 15\mathbf{w} + (-4\mathbf{u} + 3\mathbf{v} + \mathbf{w}) = 14\mathbf{u} - 16\mathbf{v} + 16\mathbf{w}$$

(Note: this is exactly the proof of Theorem 12.7 applied to the present example).

3. A line perpendicular to the tangent line of a plane curve is called a normal line. If the normal line and a horizontal line are drawn at any point of a certain plane curve C, they cut off a segment of length 2 on the y-axis. Write down the cartesian equations of all curves C with the above property, assuming that they pass trough the point $(2, 1)$.

Solution. This is exercise 14 of §14.9 p. 529 Vol. I (solved in class. You can also find a solution in some exams of previous years) with the role of the x-axis and of the y-axis exchanged. The curves with the required properties are two parabolas having the y-axis as symmetry axis.

4. Let V denote the linear space of all real polynomials of degree ≤ 2. Let $T : V \to V$ be the linear transformation defined as follows: $T(a+bx+cx^2) = a+b+b(x+1)+c(x+1)^2$.
(a) Given a polynomial $P(x) = c + dx + ex^2 \in V$, write down a formula for $T^{-1}(P(x))$.
(b) Is T diagonalizable?

Solution. (a) After an immediate calculation we can write $T(a + bx + cx^2 = a + 2b + c + (b + 2c)x + cx^2$. Therefore, with respect to the standard basis $\mathcal{E} = \{1, x, x^2\}$,

$$m_{\mathcal{E}}^{\mathcal{E}}(T) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

We know that $m_{\mathcal{E}}^{\mathcal{E}}(T^{-1}) = \left(m_{\mathcal{E}}^{\mathcal{E}}(T)\right)^{-1}$. Therefore we have to find the inverse of the matrix $m_{\mathcal{E}}^{\mathcal{E}}(T)$. Computing, one finds that the inverse is $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore

$$T^{-1}(c + dx + ex^2) = c - 2d + 3c + (d - 2e)x + ex^2$$

(b) The answer is: NO. Indeed, from the matrix $m_{\mathcal{E}}^{\mathcal{E}}(T)$ one sees immediately that T has $\lambda = 1$ as triple eigenvalue, but, clearly, the corresponding eigenspace $E(1)$ isn't three-dimensional.

5. Let $\mathbf{u} = (1, 1, -1)$, $\mathbf{v} = (2, 1, -1)$, $\mathbf{w} = (3, 2, -1)$.
Let $T : V^3 \to V^3$ defined as follows:
$T(\mathbf{u}) = 2\mathbf{u} = (2, 2, -2)$, $T(\mathbf{v}) = -3\mathbf{v} = (-6, -3, 3)$, $T(\mathbf{w}) = 2\mathbf{u} - 3\mathbf{v} = (-4, -1, 1)$.
(a) Compute dimension and a basis of $N(T)$.

(b) Compute eigenvectors and eigenspaces of T. If possible, find a basis $\mathcal{B}$ of V^3 and a diagonal matrix D such that $m_{\mathcal{B}}^{\mathcal{B}}(T) = D$.

Solution. (a) Clearly $T(V^3)$ is two dimensional (in fact $T(V^3) = L(\mathbf{u}, \mathbf{v})$). Therefore $\dim N(T) = 3 - 2 = 1$. To find a generator one can solve a system. However, there is no need for that: since $T(\mathbf{w}) = T(\mathbf{u}) + T(\mathbf{v})$, by linearity we see that $T(\mathbf{u} + \mathbf{v} - \mathbf{w}) = O$. Therefore $N(T) = L(\mathbf{u} + \mathbf{v} - \mathbf{w})$.
(b) This case is very simple: from the definition of T we see that:
- 2 is an eigenvalue (of eigenvector $\mathbf{u}$);
- -3 is an eigenvalue (of eigenvector $\mathbf{v}$);
- From point (a) we have also that 0 is an eigenvalue (of eigenvector $\mathbf{u} + \mathbf{v} - \mathbf{w}$)). Indeed $T(\mathbf{u} + \mathbf{v} - \mathbf{w}) = O = 0(\mathbf{u} + \mathbf{v} - \mathbf{w})$.

Therefore T, having three distinct real eigenvalues, is diagonalizable, and, taking $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v} - \mathbf{w})\}$, we have that

$$m_{\mathcal{B}}^{\mathcal{B}}(T) = diag(2, -3, 0)$$

Alternatively, one can diagonalize the matrix $m_{\mathcal{C}}^{\mathcal{C}}(T) = \begin{pmatrix} 2 & 0 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$, where

$\mathcal{C} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.