L. A. G. **final exam n.1,** **june 27, 2016** **Name:**

1. Let $P = (1,0,0)$ and, for a varying in $\mathbf{R}$, $A_a = (1, a, -1)$. For a varying in $\mathbf{R}$ let us consider the line of V_3: $S_a = \{\, P + sA_a \mid s \in \mathbf{R}\,\}$.
Let L be the following line of V_3: $L = \{(9,-3,-1) + t(1,0,-1) \mid t \in \mathbf{R}\}$.
(a) Find all $a \in \mathbf{R}$ such that there is a plane containing both S_a and L.
(b) For such values of a write down the parametric equation of the plane.

Solution. Known fact to be used: there is a plane containing both lines if and only if the two lines are parallel or they meet. It is easy to see that, for all a, they don't meet (try to find the intersection point and you'll see that the system has no solution for all a). However, the two lines can be parallel: this happens if and only if $a = 0$. In this case the plane containing them is

$$\Pi = P + tA_0 + s((9,-3,-1) - P) = (1,0,0) + t(1,0,-1) + s(8,-3,-1) \quad t, s \in \mathbf{R}.$$

2. Let L be the line of V_2 whose cartesian equation is $4x + 3y = 10$.
Moreover let $\mathcal{X}$ be the subset of V_2 whose elements are the pairs (x, y) such that

$$2 \parallel (x,y) \parallel = d\big((x,y)), L\big).$$

(a) Find the maximum of the norms of elements of $\mathcal{X}$ and find all elements of $\mathcal{X}$ having maximal norm. Find the minimum of the norms of elements of $\mathcal{X}$ and find all elements of $\mathcal{X}$ having minimal norm.
(b) Draw a rough sketch of $\mathcal{X}$.

Solution. The condition defining $\mathcal{X}$ can be written as

$$d((x,y), O) = \frac{1}{2} d((x,y), L)$$

Therefore $\mathcal{X}$ is the ellipse of eccentricity $e = \frac{1}{2}$, with one focus at the origin and the corresponding directrix equal to the line L. The points of maximal norm and minimal norm are the two vertices. To find the two vertices one can use the polar equation. We have $d = d(O, L) = \frac{10}{5} = 2$. The unit normal vector to L is $N = \frac{1}{5}(4,3)$. The focus O is in the negative half-plane. Therefore the polar equation is

$$r = \frac{\frac{1}{2}2}{1 + \frac{1}{2}cos\phi}$$

where ϕ is the angle between (x, y) and N. The minimum (resp. the maximum) occur for $\phi = 0$ (resp. $\phi = \pi$), that is the two vertices, as we know. The minimal norm is $\frac{2}{3}$. The maximal norm is 2. The point of minimal distance is $\frac{2}{3}N$. the point of maximal distance is $-2N$.

3. A point moves in V_3 with position vector $\mathbf{r}(t)$, in such a way that its acceleration vector $\mathbf{a}(t)$ is parallel to the vector $\mathbf{r}(t) - (1, 2, -1)$ for all t. Moreover, at the initial time $t = 0$ we have that $\mathbf{r}(0) = (-1, 3, 2)$ and $\mathbf{r}'(0) = (1, 1, 1)$.
Is there a plane containing $\mathbf{r}(t)$ for all t? (that is: is the trajectory of the point is contained in a plane?)
If the answer is no, find an example of such a motion such that the trajectory of the point is not contained in any plane. If the answer is yes, find the equation of the plane, and explain why the trajectory is contained in that plane.

Solution. The answer is: yes, the trajectory of the point is contained in a plane. It is a calculation that we did more than one time (see for example Ex. 10 of Section 14.9, there are also similar exercises). The condition of the motion is described by the equation

$$(\mathbf{r} - P)'' \times (\mathbf{r} - P) \equiv 0$$

where $P = (1, 2, -1)$. This can be also written

$$\big((\mathbf{r} - P)' \times (\mathbf{r} - P)\big)' \equiv 0$$

that is

$$(\mathbf{r} - P)' \times (\mathbf{r} - P) \equiv \mathit{constant} := \mathbf{u}$$

Therefore, since $\mathbf{r}(t) - P$ is always perpendicular to $\mathbf{u}$, the trajectory is contained in the plane of cartesian equation

$$(X - P) \cdot \mathbf{u} = 0$$

The constant vector $\mathbf{u}$ can be computed using the initial conditions:
$\mathbf{u} = (\mathbf{r}'(0) - P) \times (\mathbf{r}(0) - P)$.

4. Let V be the linear space of real polynomials of degree ≤ 2, equipped with the inner product:

$$(P, Q) = P(0)Q(0) + P'(0)Q'(0) + P''(0)Q''(0)$$

Find two polynomials P and Q in V such that $x^2 = P(x) + Q(x)$ and satisfying the following properties:
- $P(-1) = 0$
- Q is perpendicular to all polynomials $F \in V$ such that $F(-1) = 0$. *(Hint: before embarking on calculations, understand the formula defining the inner product).*

Solution. For a polynomial $P(x) = a_0 + a_1 x + a_2 x^2$ we have that $P(0) = a_0$, $P'(0) = a_1$, $P''(0) = 2a_2$. Therefore, letting $Q(x) = b_0 + b_1 x + b_2 x^2$, we have that

$$(P(x), Q(x)) = a_0 b_0 + a_1 b_1 + 4a_2 b_2$$

The exercise asks for the orthogonal decomposition of the polynomial x^2 as sum of an element of $W = \{P(x) \in V \mid P(-1) = 0\}$ and of an element of $W^\perp$:

$$x^2 = p_W(x^2) + P_{W^\perp}(x^2)$$

Since W is two-dimensional, $W^\perp$ is one-dimensional, hence it is shorter to compute $P_{W^\perp}(x^2)$.

First we compute $W^\perp$: a basis of W is, for example $\{x+1, x^2+x\}$. Therefore $W^\perp$ is the linear space of all polynomials $Q(x) = a_0 + a_1 x + a_2 x^2$ perpendicular to both $x+1$ and x^2+x. Using our formula for the inner product, we get the system $\begin{cases} a_0 + a_1 = 0 \\ a_1 + 4a_2 = 0 \end{cases}$. Therefore

$$W^\perp = L(4 - 4x + x^2)$$

Hence

$$Q(x) = p_{W^\perp}(x^2) = \frac{(x^2, 4-4x+x^2)}{(4-4x+x^2, 4-4x+x^2)}(4-4x+x^2) = \frac{4}{9}(4-4x+x^2) \quad \text{and}$$

$$P(x) = x^2 - Q(x) = \frac{4}{9}(-1 + x + 2x^2)$$

5. Let U be a linear subspace of V_4 Let $T_U : V_4 \to V_4$ be the linear transformation defined as follows: $T_U(X) = P_U(X) + X$ (where P_U denotes the projection onto U).
(a) Find eigenvalues and eigenspaces of T_U. Is T_U diagonalizable?
(b) Compute the null-space and range of T_U.
(c) Now let let $U = L((1,0,1,1),(1,1,-1,-1))$. . Find (if possible) a basis $\mathcal{B}$ of V_4 such that the representing matrix $m^{\mathcal{B}}_{\mathcal{B}}(T_U)$ is diagonal.

Solution. (a) For $X \in U$, we have that $T_U(X) = P_U(X) + X = X + X = 2X$. Therefore 2 is an eigenvalue and U is contained in the eigenspace of 2. For $X \in U^\perp$, $T_U(X) = P_U(X) + X = O + X = X$. Therefore 1 is an eigenvalue and $U^\perp$ is contained in the eigenspace of 1. Putting absis of U and a basis of $U^\perp$ together we find that T_U is diagonalizable and that 2 and 1 are the only eigenvalues of T_U
(b) The nullspace of T is $\{O\}$: in fact, if it was bigger that that, 0 would be an eigenvalue, which we know is not true. Therefore, by the nullity plus rank theorem, $T_U(V_4) = V_4$.
(c) One has to find a basis of $U^\perp$ (it is a two-dimensional linear subspace). Putting together with the basis of U we have a basis $\mathcal{B}$ of V_4 such that $m^{\mathcal{B}}_{\mathcal{B}}(T_U) = diag(2,2,1,1)$.