L. A. G. final exam n.7, feb 16, 2017 Name:

1. For λ varying $\mathbf{R}$, let us denote L_λ the following line of V_3: $\begin{cases} x+z=0 \\ 3x+y+(\lambda-1)z=1 \end{cases}$.

Moreover, let M be the line of V_3: $\begin{cases} x+y=1 \\ 2x+y+3z=1 \end{cases}$.

Find for which $\lambda \in \mathbf{R}$ the lines L_λ and M meet, and, for such values of λ, compute the intersection point.

Solution. The final answer is:
(a) The lines L_λ and M meet for every $\lambda \in \mathbf{R}$;
(b) The intersection point $L_\lambda \cap M$ is $(0,1,0)$ for every $\lambda \in \mathbf{R}$.

To get the solution one can (for example) consider the system obtained putting together the two systems defining L_λ and M:

$$\begin{cases} x+z=0 \\ 3x+y+(\lambda-1)z=1 \\ x+y=1 \\ 2x+y+3z=1 \end{cases}$$

This correspond to the augmented matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 1 & \lambda-1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 \end{pmatrix}$.

With gaussian elimination we obtain

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 1 & \lambda-1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \lambda-4 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & \lambda-4 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \lambda-5 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

The last matrix corresponds to the system (equivalent to the initial one):

$$\begin{cases} x+z=0 \\ y+z=1 \\ (\lambda-5)z=0 \\ -2z=0 \end{cases}$$

which is compatible for every λ, and has always the unique solution $(0,1,0)$.

2. Let C be the ellipse whose vertices are $(4,3)$ and $(14,3)$, and containing the point $(9,6)$.
(a) Write down the cartesian equation of C
(b) Find the points of C whose curvature is minimal, and compute the curvature at those points.

Solution. Test 5, 2014 (in the file it is named test 4, 2014).

3. Let V be the space of all real polynomials of degree ≤ 3, equipped with the following inner product:

$$< P(x), Q(x) >= P(0)Q(0) + P'(0)Q'(0) + P''(0)Q''(0) + P(1)Q(1)$$

Let us consider the linear subspace $W = L(1, x, x^2)$. Compute the distance between $P(x) = x^3$ and W. Compute the element of W which is closest to $P(x) = x^3$.

Solution. Given a polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, we have clearly that $P(0) = a_0$, $P'(0) = a_1$, $P''(0) = 2a_2$ and $P(1) = a_0 + a_1 + a_2 + a_3$. Therefore, given also the polynomial $Q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ we have that

$$< P(x), Q(x) >= a_0 b_0 + a_1 b_1 + 4a_2 b_2 + (a_0 + a_1 + a_2 + a_3)(b_0 + b_1 + b_2 + b_3) \tag{1}$$

Now we have an explicit formula for

$$< P(x), Q(x) >$$

. In order to solve the exercise, there are two ways:
(1) Compute the projection of x^3 on W. In order to do so, we have to compute previously an orthogonal basis of W using the Gram-Schmidt process;
(2) Compute the projection of x^3 on $W^\perp$. In order to do so, we have to previously understand who is $W^\perp$.

Note that the ambient space V is 4-dimensional, and the subspace W is 3-dimensional. Therefore $W^\perp$ is 1-dimensional. Therefore, in order to minimize the calculations, it seems (slightly) more convenient to use method (2).

Let $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. We have that $P(x) \in W^\perp$ if and only if $\begin{cases} < P(x), 1 >= 0 \\ < P(x), x >= 0 \\ < P(x), x^2 >= 0 \end{cases}$

Using the formula (1) this means

$$\begin{cases} a_0 + (a_0 + a_1 + a_2 + a_3) = 0 \\ a_1 + (a_0 + a_1 + a_2 + a_3) = 0 \\ 4a_2 + (a_0 + a_1 + a_2 + a_3) = 0 \end{cases} \quad \text{that is} \quad \begin{cases} 2a_0 + a_1 + a_2 + a_3 = 0 \\ a_0 + 2a_1 + a_2 + a_3 = 0 \\ a_0 + a_1 + 5a_2 + a_3 = 0 \end{cases}$$

Solving with gaussian elimination we have

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 5 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & 9 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 26 & 2 \end{pmatrix}$$

which has as space of solutions (say calculation) $L((4, 4, 1, -13))$. Therefore

$$W^\perp = L(4 + 4x + x^2 - 13x^3)$$

Now we compute the projection

$$p_{W^\perp}(x^3) = \Big(\frac{< x^3\, ,\, 4+4x+x^2-13x^3 >}{< 4+4x+x^2-13x^3\, ,\, 4+4x+x^2-13x^3 >} \Big) (4+4x+x^2-13x^3) =$$

$$= -\frac{4}{52}(4+4x+x^2-13x^3) = -\frac{1}{13}(4+4x+x^2-13x^3)$$

Therefore the distance between x^3 and W is

$$d(x^3, W) = \parallel p_{W^\perp}(x^3) \parallel = \parallel -\frac{1}{13}(4+4x+x^2-13x^3) \parallel = \frac{\sqrt{52}}{13}$$

and the closest element is

$$p_W(x^3) = x^3 - p_{W^\perp}(x^3) = -4 - 4x - x^2 + 12x^3$$

4. Consider the following real quadratic form $Q : \mathcal{V}_4 \to \mathbf{R}$,
(b) $Q(x, y, z, t) = x^2 + 2xy + 2xz + 2xt + y^2 + 2yz + 2yt + z^2 + 2zt + t^2$.

Find a canonical form for Q. In other words, find an orthonormal basis $\mathcal{B}$ of V_4 and four real numbers λ_i, $i = 1, 2, 3, 4$ such that

$$Q(x, y, z, t) = \lambda^1 (x')^2 + \lambda^2 (y')^2 + \lambda^3 (z')^2 + \lambda^4 (t')^2$$

where (x', y', z', t') is the vector of components of (x, y, z, t) with respect to the basis $\mathcal{B}$.

Solution. Exercise (b) fifth session 2012-'13.