L. A. G. 2015. Fifth intermediate test, june 3, 2015

1. With the inner product: $(P,Q) = \int_{-3}^{3} P(x)Q(x)dx$, find the polynomial of degree ≤ 2 that is nearest to the polynomial $x^3 - 1$.

Solution. Let V be linear space of alla polynomials and $W \subset V$ be the space of polynomials of degree ≤ 2. Let $P(x) = x^3 - 1 \in V$. The polynomial in W which is nearest to $P(X)$ is $pr_W(P(x))$ (the projection of $P(x)$ onto W).

In order to compute that, we first need an orthogonal basis of W. We have at our disposal the standard basis $\{1, x, x^2\} = \{U_1(x), U_2(x), U_3(x)\}$, which is not orthogonal, so we need to orthogonalize it. We apply the Gram-Schmdt method. We have that $U_1(x)$ and $U_2(x)$ are already orthogonal. The third vector is $V_3(x) = x^2 - (\frac{(x^2,1)}{(1,1)})1 - (\frac{(x^2,x)}{(x,x)})x = x^2 - 3$. Therefore an orthogonal basis of$W$ is $\{1, x, x^2 - 3\}$. The projection of $x^3 - 1$ over W is the sum of the projections of $x^3 - 1$ onto the lines spanned by the three vectors of the orthogonal basis:

$$pr_W(x^3 - 1) = (\frac{(x^3 - 1, 1)}{(1,1)})1 + (\frac{(x^3 - 1, x)}{(x,x)})x + (\frac{(x^3 - 1, x^2 - 3)}{(x^2 - 3, x^2 - 3)})(x^2 - 3) =$$

$$= -1 + \frac{\frac{2 \cdot 3^5}{5}}{18}x + 0$$