L. A. G. 2015. Fourth intermediate test, may 13, 2015

Solve two of the following exercises (at your choice).
Only after having completed them, it you have time, you can try to to solve also the remaining one.
Each exercise is worth of 0.5 points.

1. A point moves according to the function $\mathbf{r}(t) = (3t^2, 2t^3, 3t)$. (a) For $t = 1$ find the unit tangent vector, the unit normal vector, and express the acceleration vector as linear combination of them. (b) Compute the curvature of the underlying curve at the point $\mathbf{r}(1)$.

2. A point moves on the ellipse of equation $\frac{2x^2}{3} + \frac{y^2}{6} = 1$ with position vector $\mathbf{r}(t) = (x(t), y(t))$ in such a way that $y'(t) = 3x(t)$ for each t. Find $\mathbf{r}(t)$ knowing that $\mathbf{r}(0) = (0, -\sqrt{6})$

3. Given a point P on a curve in $\mathcal{V}_2$, denote N_P the intersection of the normal line (at P) with the x-axis.
A curve $\mathcal{C}$ in $\mathcal{V}_2$ has the following property: $d(O, P) = d(N_P, P)$ for each point $P \in \mathcal{C}$ (where O denotes the origin). Moreover, the curve passes through $(3, 1)$. Describe the curve and find its cartesian equation.

SOLUTION

1. $\underline{\mathbf{v}}(1) = (6, 6, 3)$, $T(1) = \frac{1}{9}(6, 6, 3) = \frac{1}{3}(2, 2, 1)$, $\underline{\mathbf{a}}(1) = (6, 12, 0)$. To answer the question, we project $\underline{\mathbf{a}}(1)$ onto $\underline{\mathbf{v}}(1)$, so that we decompose $\underline{\mathbf{a}}(1)$ as the sum of a vector parallel to $\underline{\mathbf{v}}(1)$ and a vector perpendicular to $\underline{\mathbf{v}}(1)$:

$$(6, 12, 0) = (8, 8, 4) + (-2, 4, -4) = 12\,(\frac{1}{3}(2, 2, 1)) + 6(\frac{1}{3}(-1, 2 - 2))$$

It follows that the unit normal vector $N(1)$ is $\frac{1}{3}(-1, 2 - 2)$ and the required decomposition is

$$\underline{\mathbf{a}}(1) = (6, 12, 0) = 12\,(\frac{1}{3}(2, 2, 1)) + 6(\frac{1}{3}(-1, 2 - 2)) = 12\,T(1) + 6\,N(1)$$

2. We know that $\frac{2x(t)^2}{3} + \frac{y(t)^2}{6} = 1$. Differentiating we get $4x(t)x'(t) + y(t)y'(t) = 0$. Plugging the condition $y'(t) = 3x(t)$ we get $4x(t)x'(t) + 3y(t)x(t) = 0$ which implies

$x'(t) = -\frac{3}{4}y(t)$. Differentiating, we get

$$\begin{cases} x''(t) = -\frac{3}{2}x(t) \\ y''(t) = -\frac{3}{2}y(t) \end{cases}$$

It follows, using what we know about the differential equations of the form $f'' = \lambda f$, with $\lambda < 0$, that both $x(t)$ and $y(t)$ are linear combinations of the functions $\cos\frac{3}{2}t$ and $\sin\frac{3}{2}t$.

Let us find the scalars of both linear combinations using the initial condition. Using that $x(0) = 0$ we get that $x(t) = c\sin\frac{3}{2}t$, for $c \in \mathbf{R}$.
Now $y(t)$ is of the form $d\cos\frac{3}{2}t + e\sin\frac{3}{2}t$. Hence $y'(t) = \frac{3}{2}(-d\sin\frac{3}{2}t + e\cos\frac{3}{2}t)$. Therefore $y'(0) = -\frac{3}{2}e = 3x(0) = 0$ whence $e = 0$. Hence $y(t) = d\cos\frac{3}{2}t$. Since the hypothesis says that $y(0) = \sqrt{6}$ we get $d = \sqrt{6}$. Finally, since $x'(0) = \frac{3}{2}c = -\frac{3}{4}y(0) = -\frac{3}{4}\sqrt{6}$, we get $c = -\frac{1}{2}\sqrt{6}$. In conclusion

$$\mathbf{r}(t) = (-\frac{1}{2}\sqrt{6}\sin\frac{3}{2}t, \sqrt{6}\cos\frac{3}{2}t)$$

3. Let $\mathbf{r}(t) = (x(t), y(t))$ be a parametrization of the curve. At the point $P(t) = (x(t), y(t))$ the normal line is:

$$\{(x(t), y(t)) + \lambda(-y'(t), x'(t)) \mid \lambda \in \mathbf{R}\}.$$

To compute its intersection with the x-axis we set $y = 0$ in the parametric equation, so that $y(t) + \lambda x'(t) = 0$, hence $\lambda = -\frac{y(t)}{x'(t)}$. Hence the intersection with the x-axis is

$$(\, x(t) - \frac{y(t)}{x'(t)}(-y'(t))\,,\, 0) = (x(t) + \frac{y(t)y'(t)}{x'(t)}, 0).$$

The squared distance between this point and $P(t)$ is $\frac{(y'(t))^2(y(t))^2}{(x'(t))^2} + y(t)^2$. Obviously the squared distance between $P(t)$ and the origin is $(x(t))^2 + (y(t))^2$. Therefore the hypothesis is

$$|\frac{y'(t)y(t)}{x'(t)}| = |x(t)|$$

that is

$$|yy'| = |xx'|$$

Therefore, getting rid of the absolute values and integrating, we get the two possible cartesian equations

$$x^2 - y^2 = constant \qquad \text{or} \qquad x^2 + y^2 = constant$$

The second one is a circle centered at the origin. In this case the point $N_P \equiv O$. Imposing the passage through $(3,1)$ we get that the constant (that is, the square of the radius) is 10.

The first one is a hyperbola. Imposing the passage through $(3,1)$ we get that the constant is 8. Therefore the cartesian equation is

$$x^2 - y^2 = 8$$