L. A. G. 2015. third intermediate test, april 22, 2015

Let us consider the conic section containing the point $(-4, -3)$, with one focus at the origin and corresponding directrix of cartesian equation $4x - 3y = \frac{11}{2}$. Find:
(a) eccentricity; (b) vertices; (c) center of symmetry; (d) the other focus.

Solution

(a) *Eccentricity* To begin with, the knowledge of a point and of a directrix of the conic section allows us to compute its eccetricity: the distance between $(-4, -3)$ and the directrix, which will be denoted L_1, is $5/2$. On the other hand, as the corresponding focus F_1 is the origin, $d((-4, -3), F) = \| (-4, -3) \| = 5$.
Since, by definition, $d((-4, -3), F_1) = e\, d((-4, -3), L_1)$, we get $e = 2$. Hence the conic section is a hyperbola.

(b) *Vertices* Since the focus is in the negative half-plane determined by the unit normal vector

$$N = \frac{1}{5}(4, -3)$$

(indeed $4 \cdot 0 - 3 \cdot 0 = 0 < 11/2$), we have the polar equations of the two branches

$$r = \frac{ed}{e\, \cos\varphi + 1} \qquad \text{and} \qquad r = \frac{ed}{e\, \cos\varphi - 1}$$

where $d = d(F, L_1) = 11/10$ and φ is the angle between a point of the conic section and the unit vector N. As the vertices are on the line passing trough the focus and parallel to N, we have that for the vertices $\varphi = 0$ oir $\varphi = \pi$. In fact it is easily seen that in both equations only $\varphi = 0$ is admissible. Hence

$$r = \frac{2(11/10)}{3} = \frac{11}{15} \qquad \text{and} \qquad r = \frac{2(11/10)}{1} = \frac{11}{5}$$

Therefore, calling V_1 and V_2 the two vertices, we get

$$V_1 = \frac{11}{15}N \qquad \text{and} \qquad V_2 = \frac{11}{5}N$$

(c) *Center of symmetry* The center of symmetry, say C, is the half-point between V_1 and V_2. Therefore

$$C = \frac{22}{15}N$$

(d) *The other focus* The other focus, say F_2 is on the line spanned by N, and symmetric to F_1 with respect to C. hence

$$F_2 = \frac{44}{15}N$$