Linear Algebra and Geometry. Intermediate test 1, march 18, 2015.

1. Let $A = (1, -1, -1)$ and $B(1, 2, 3)$.
(a) Find a vector C parallel to A and a vector D perpendicular to A such that

$$2A - B = C + D$$

(b) Find all vectors parallel to A whose norm is 9.

Solution

(a) What is required in the ortogonal decomposition of the vectore $2A - B$ in a vector parallel to A and a vector perperdicular to A. Hence

$$C = \left(\frac{(2A - B) \cdot A}{A \cdot A}\right)A = \frac{10}{3}(1, -1, -1)$$

and

$$D = (2A - B) - C = \frac{1}{3}(-7, -2, -5)$$

(b) We look for all $c \in \mathbf{R}$ such that $\| cA \| = 9$. But $\| cA \| = |c| \, \| A \| = |c|\sqrt{3}$. Therefore $|c| = \frac{9}{\sqrt{3}} = 3\sqrt{3}$. Hence there are two solutions $c = 3\sqrt{3}$ and $c = -3\sqrt{3}$. In conclusion, there are two vectors as requested:

$$3\sqrt{3}\,A = 3\sqrt{3}\,(1, -1, -1) \qquad \text{and} \qquad -3\sqrt{3}\,A = -3\sqrt{3}\,(1, -1, -1)$$

2. Let $A = (1, 1, 1)$ and $B = (2, 0, 1)$
(a) Give an example of a vector $C \in V_3$, non-parallel to A and B, such that $\{A, B, C\}$ is a linearly dependent set.
(b) Is there a vector $D \in V_3$ such that $\{A, B, D\}$ is a linearly dependent set? (justify your answer). If the answer is yes, provide an example of such a vector D.

Solution

(a) Any linear combination of A and B with both coefficients non-zero works. For example $C = A + B$. In this case we have $A + B - C = O$.
(b) The answer is yes, because we know by a Theorem (to be precise: Theorem 12.10(b)) that $\{A, B\}$ is contained in some bases of V_3, that is, in some linearly independent set of the form $\{A, B, D\}$. Note that such D's are precisely those wich are not contained in the linear span $L(A, B)$. To find concretely such a D the shortest ways are:
(1) try with the unit vectors of the coordinate axes. For example, if one tries with $D = (1, 0, 0)$, it is easily checked that $\{A, B, D\}$ is an independent set.
(2) Take a vector D perpendicular to both A and B (then we know that $\{A, B, D\}$ is an independent set). For example: $D = (1, 1, -2)$.