L. A. G. 2015, nov 27, 2015

1. In V^4 (equipped with the usual dot product) let $\mathbf{v} = (1, -1, 2, 3)$. Among all vectors (x, y, z, t) such that $x - y + z - t = 0$ find the one which is closest to $\mathbf{v}$, and compute its distance from $\mathbf{v}$.

Solution. The set of solutions of the equation $x - y + z - t = 0$ is a linear space, say W. The vector of W which is closest to $\mathbf{v}$ is nothing else than $pr_W(\mathbf{v})$, the projection of $\mathbf{v}$ onto W. We know that $\mathbf{v} = pr_W(\mathbf{v}) + pr_{W^\perp}(\mathbf{v})$. Since $W^\perp$ is 1-dimensional (note that $W^\perp = L((1, -1, 1, -1))$, it is simpler to compute $pr_{W^\perp}(\mathbf{v})$ and then compute

$$pr_W(\mathbf{v}) = \mathbf{v} - p_{W^\perp}(\mathbf{v}) = (1, -1, 2, 3) - \frac{1}{4}(1, -1, 1, -1) = \frac{1}{4}(3, -3, 7, 13)$$

Its distance from $\mathbf{v}$ is $\| \frac{1}{4}(1, -1, 1, -1) \| = \frac{2}{4} = \frac{1}{2}$.

2. A point moves in space is such a way that, at time t_0, its position vector is $(3, 5, -3)$, its velocity vector is $(1, 2, -2)$ and its acceleration vector is $(2, 2, 0)$. At time t_0 find: the unit normal vector, the equation of the osculating plane, the radius of curvature, the center of curvature, a parametrization of the osculating circle.

Solution. - Osculating plane: it is the plane $\mathbf{r}(t_0) + \{s\mathbf{v}(t_0) + \lambda \mathbf{a}(t_0) \mid s, \lambda \in \mathbf{R}\}$.
- Unit normal vector: by means of the orthogonal projection, we decompose $\mathbf{a}(t_0)$ as the sum of a vector parallel to $\mathbf{v}(t_0)$ and a vector perpendicular to $\mathbf{v}(t_0)$ and we express the sum as a linear combination of unit vectors:

$$(2, 2, 0) = \frac{6}{9}(1, 2, -2) + \big((2, 2, 0) - \frac{2}{3}(1, 2, -2)\big) = 2\big(\frac{1}{3}(1, 2, -2)\big) + \frac{1}{3}\big((4, 2, 4)\big) =$$

$$2(\frac{1}{3}(1, 2, -2)) + 2\big(\frac{1}{6}(4, 2, 4)\big)$$

By unicity of the orthogonal decomposition, this must coincide with the decomposition:

$$\mathbf{a}(t_0) = v'(t_0)T(t_0) + v(t_0) \parallel T'(t_0) \parallel N(t_0).$$

Therefore

$$N(t_0) = \frac{1}{6}(4, 2, 4)$$

- Radius of curvature: it is the inverse of the curvature which is computed by the usual formula $\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{v^3}$. However, since we know that $\kappa = \frac{\|T'\|}{v}$ and, from the above decomposition, $v = 3$ and $v \parallel T' \parallel = 2$, we get that

$$\kappa = 2\,\frac{1}{v^2} = \frac{2}{9}$$

Therefore the radius of curvature at t_0 is

$$R(t_0) = \frac{9}{2}.$$

- Center of curvature at t_0: it is

$$C(t_0) = \mathbf{r}(t_0) + R(t_0)N(t_0) = (3, 5, -3) + \frac{9}{2} \cdot \frac{1}{6}(4, 2, 4) = (3, 5, -3) + \frac{3}{4}(4, 2, 4)$$

- Osculating circle at t_0: it is the circle whose center in the center of curvature, and passing through $\mathbf{r}(t_0)$, that is:

$$C(t_0) + R(t_0)\big(\cos\theta(\mathbf{r}(t_0) - N(t_0)) + \sin\theta(T(t_0))\big), \quad \theta \in [0, 2\pi]$$

3 . Let V be the linear space of real polynomials of degree ≤ 2. For each of the following functions, establish whether they are linear transformations. For those which are linear transformations, compute their representative matrices with respect to the basis $\mathcal{E} = \{1, x, x^2\}$ and establish whether they are *invertible* linear transformations:
(a) $T : V \to V$, $T(P(x)) = xP'(x+1) - (x+1)P'(x)$;
(b) $S : V \to V$, $S(P(x)) = xP'(x+1) - P(x) - x$;
(c) $R : V \to V$, $R(P(x)) = (x^2 + x + 1)P''(x+1) - 2P(x)$

Solution. (a) $T(P(x) + Q(x)) = x(P(x+1) + Q(x+1))' - (x+1)(P(x) + Q(x)) = xP'(x+1) - (x+1)P'(x) + xQ'(x+1) - (x+1)Q'(x) = T((P(x)) + T(Q(x))$.
$T(\lambda P(x)) = x\lambda P'(x+1) - (x+1)\lambda P'(x) = \lambda T(P(x))$.
Therefore T is linear. The representative matrix is $m_{\mathcal{E},\mathcal{E}}(T) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is not an invertible matrix. Therefore T is not an invertible linear transformation.
(b)

$$S(P(x) + Q(x)) = x(P(x+1) + Q(x+1))' - (P(x) + Q(x)) - x =$$
$$xP'(x+1) - P(x) + xQ'(x+1)) - Q(x) - x$$
$$S(P(x)) + S(Q(x)) = xP'(x+1) - P(x) - x + xQ'(x+1)) - Q(x) - x =$$
$$xP'(x+1) - P(x) + xQ'(x+1)) - Q(x) - 2x.$$

Therefore S is *not* linear.
(c) $R(P(x) + Q(x)) = (x^2 + x + 1)(P(x+1) + Q(x+1))'' - 2(P(x) + Q(x)) = (x^2 + x + 1)P''(x+1) - 2P(x) + Q''(x+1) - 2Q(x) = R(P(x)) + R(Q(x))$.
$R(\lambda P(x)) = (x^2+x+1)\lambda P''(x+1) - 2\lambda P(x) = \lambda((x^2+x+1)P''(x+1) - 2P(x)) = \lambda R(P(x))$.
therefore R is linear.
The representative matrix is $m_{\mathcal{E},\mathcal{E}}(R) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. It is not an invertible matrix.
Therefore R is not an invertible linear transformation.

4. Let $T : V^3 \to V^3$ the linear transformation defined by $T(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} -x + 5y - 2z \\ -x - y + z \\ 5x - y - 2z \end{pmatrix}$

.

(a) Is T one-to-one? If the answer is in the affirmative, explain why. If the answer is in the negative, find two different vectors $\mathbf{u}, \mathbf{v} \in V^3$ such that $T(\mathbf{u}) = T(\mathbf{v})$.
(b) Is T surjective? If the answer is in the affirmative, explain why. If the answer is in the negative show a vector $\mathbf{w}$ which does not belong to $T(V^3)$.

Solution. (a) Computing the null-space $N(T)$ we get $N(T) = L((9, 1, 2))$. Therefore T is not injective. For example, $T((0, 0, 0)) = T((9, 1, 2)) = (0, 0, 0)$. More generally, any two vectors $\mathbf{u}$, $\mathbf{v}$ such that $\mathbf{u} - \mathbf{v} \in N(T)$ have the same value via T.
(b) We have that $rk(T) = 2$. Therefore T is not surjective. Since $T(V^3) = L((-1, -1, 5), (5, -1, -1))$, which has cartesian equation $x + 4y + z = 0$, a vector which does not satisfy that equation does not belong to the range of T. For example: $\mathbf{w} = (1, 0, 0)$.

5. Compute (in the usual coordinates (x, y)) the center/vertex, foci/focus and directrices/directrix of the conic section of cartesian equation: $x^2 + 2xy + y^2 + 2x - 2y + \frac{1}{\sqrt{2}}$.

Solution. The quadratic part is $Q(x, y) = x^2 + 2xy + y^2$. The matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ whose eigenvalues are 0 and 2. Correspondingly, we get the orthonormal basis of eigenvectors $\mathcal{B} = \{\frac{1}{\sqrt{2}}(1, -1), \frac{1}{\sqrt{2}}(1, 1)\}$. The change of basis matrix is $M_{\mathcal{B},\mathcal{E}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The components of (x, y) with respect to the basis $\mathcal{B}$, say (x', y'), are related to (x, y) as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x' + y' \\ -x' + y' \end{pmatrix}$$

Substituting, we get

$$x^2+2xy+y^2+2x-2y+\frac{1}{\sqrt{2}} = 2(y')^2+2(\frac{1}{\sqrt{2}}(x'+y'))-2(\frac{1}{\sqrt{2}}(-x'+y'))+\frac{1}{\sqrt{2}} = 2(y')^2+\frac{4}{\sqrt{2}}x'+\frac{1}{\sqrt{2}} =$$

$$= 2(y')^2 + \frac{4}{\sqrt{2}}(x' + \frac{1}{4})$$

In conclusion, the equation of the conic section in the (x', y')-components is

$$(y')^2 = -\sqrt{2}(x' + \frac{1}{4})$$

The conic section is a parabola. Moreover, it follows that the (x', y')-coordinates of the vertex are $(-\frac{1}{4}, 0)$. The vertex, in the (x', y')-coordinates, is $(-\frac{1}{4}, 0) - (\frac{\sqrt{2}}{4}, 0) = (\frac{1-\sqrt{2}}{4}, 0)$. In the (x, y)-coordinates we get the vertex:

$$V = (-\frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}})$$

and the focus:

$$F = (-\frac{1-\sqrt{2}}{4\sqrt{2}}, \frac{1-\sqrt{2}}{4\sqrt{2}})$$