L. A. G. 2015. 2nd written exam, 7 - 23, 2015

1. Let us consider the plane $M = \{(1,1,0) + s(1,-1,1) + t(1,0,2) \mid s,t \in \mathbf{R}\}$ and the line $L = \{(2,0,1) + t(0,1,1) \mid t \in \mathbf{R}\}$.
(a) Is L contained in M?
(b) Find the line R contained in M, perpendicular to L, and passing through the point $(2,0,1)$.

Solution. The answer is: YES. The easiest way to check this is to write down the cartesian equation of M: $-2x - y + z = -3$, and check that $(2,0,1)$ verifies that equation, and $(0,1,1)$ verifies the associated homogeneous equation (geometrically, this means that $(0,1,1)$ is perpendicular to the normal vector $(-2,-1,1)$, hence parallel to the plane). Equivalently, one can check that all points verifying the given parametric equation of L verify also the cartesian equation of M.
(b) First of all, we need a vector which is contained in $L((1,-1,1),(1,0,2))$ and is perpendicular to $(0,1,1)$, which, by (a) belongs to $L((1,-1,1),(1,0,2))$. In general, the simplest way should be by means of the orthogonal decomposition, projecting for example $(0,1,1)$ onto $(1,-1,1)$ and taking the difference $(0,1,1) - pr_{(1,-1,1)}((0,1,1))$. In this particular case everything is especially easy, because they are already perpendicular. Therefore the required line is $\{(2,0,1) + t(1,-1,1) \mid t \in \mathbf{R}\}$.

2. A point moves in $\mathcal{V}^2$ with position vector $\mathbf{r}(t) = (t - \sin t, 1 - \cos t)$. Compute the length of the arc traced by the point from $\mathbf{r}(0)$ to $\mathbf{r}(3\pi)$.

Solution. $\mathbf{v}(t) = (1 - \cos t, \sin t)$. $v(t) = \| \mathbf{v}(t) \| = \sqrt{1 - 2\cos t + \cos^2 + \sin^2 t} = \sqrt{2}\sqrt{1 - \cos t} = \sqrt{2}\sqrt{2\sin^2(t/2)} = 2|\sin(t/2)|$ (the fourth equality follows easily from the fact that $\cos t = \cos^2(t/2) - \sin^2(t/2)$ (well known formula from trigonometry, used in MA1 to compute various integrals, and in our course to compute arc-lengths like this one). Therefore the required arc-lenght is:
$2\int_0^{3\pi} |\sin(t/2)|dt = 2(\Big[-2\cos(t/2)\Big]_0^{2\pi} + \Big[2\cos(t/2)\Big]_{2\pi}^{3\pi}) = 4(2+1) = 12$

3. Let $F : \mathcal{V}^4 \to \mathcal{V}^3$ defined by $F((x,y,z,t)) = (2x - y + z + t, x + y - z - t, -x + 2y - 2z - 2t)$.
(a) Does $(1,3,2)$ belong to $F(\mathcal{V}^4)$? If the answer is in the affirmative, describe the subset $\{(x,y,z,t) \in \mathcal{V}^4 \mid F((x,y,z,t)) = (1,3,2)\}$.
(b) Is F surjective? (This means: is $F(\mathcal{V}^4) = \mathcal{V}^3$?)

Solution. (a) $(1,3,2)$ belong to $F(\mathcal{V}^4)$ if and only if the linear system
$\begin{cases} 2x - y + z + t & =1 \\ x + y - z - t & =3 \\ -x + 2y - 2z - 2t & =2 \end{cases}$ has a solution. Solving the system one checks that this is

indeed the case. In fact, solving the system one finds that the set of solutions which is $\{(x,y,z,t) \in \mathcal{V}^4 \mid F((x,y,z,t)) = (1,3,2)\ \}$ (the preimage of $(1,3,2)$), is in fact $\{(4/3, 5/3, 0, 0) + t(0,1,1,0) + s(0,1,0,1) | t, s \in \mathbf{R}\}$. Note that $(4/3, 5/3, 0, 0)$ is a particular element of the preimage, while $\{t(0,1,1,0)+s(0,1,0,1)|t,s \in \mathbf{R}\} = L((0,1,1,0),(0,1,0,1))$ is the null-space of T, that is the premiere of $\mathbf{0}_{\mathbf{R}^3}$.
(b) Computing one finds that $rk(F) = 2$. Therefore F is not surjective, because $F(\mathcal{V}^4)$ is a 2-dimensional linear subspace of $\mathcal{V}^3$.

4. Write the polynomial x^3+1 as the sum of two polynomials, $P(x)$ and $Q(x)$, with $P(x)$ of degree ≤ 2 and $Q(x)$ such that: $\int_{-1}^{1} R(x)Q(x)dx = 0$ for all polynomials $R(x)$ of degree ≤ 2.

Solution. The question is tantamount as asking for the orthogonal decomposition of the polynomial (of degree 3) $S(x) = x^3+1$, with respect to the linear subspace $V = \{\text{polynomials of degree} \leq 2\ \}$, in the euclidean space of all polynomials (or, which works as well in this case, all polynomials of degree ≤ 3) equipped with the inner product given by $(P,Q) = \int_{-1}^{1} P(x)Q(x)dx$.

In order to find the orthogonal decomposition we first need to orthogonalize the basis $\{1, x, x^2\}$ of V. The elements 1 and x are already orthogonal, as well as x and x^2. Therefore we need only:

$$x^2 - \frac{(x^2,1)}{(1,1)}1 = \textit{computing the integrals...} = x^2 - \frac{1}{3}.$$

Therefore an orthogonal basis of V is $\{1, x, x^2 - 1/3\} := \{A, B, C\}$. The projection of $S(x) = x^3+1$ onto V is the sum of the projections on the lines spanned respectively by A, B and C. This will be the required polynomial. In practice

$$P = \frac{(S,A)}{(A,A)}A + \frac{(S,B)}{(B,B)}B + \frac{(S,C)}{(C,C)}C = \textit{very easy and short computations...} = 1 + (3/5)x$$

In conclusion, the required decomposition is

$$x^3 + 1 = (1 + (3/5)x) + (x^3 - (3/5)x)$$

5. Let $Q((x,y,z)) = -2xy - 2xz + 2yz$.
(a) Find (if possible) two perpendicular vectors $\mathbf{u}, \mathbf{v} \in \mathcal{V}^3$ such $Q(\mathbf{u}) > 0$ and $Q(\mathbf{v}) < 0$.
(b) Find (if possible) $\mathbf{w} \in \mathcal{V}^3$, $\mathbf{w} \neq \mathbf{0}$ such that $Q(\mathbf{w}) = 0$.
(c) Find a unit vector $\mathbf{r} \in \mathcal{V}^3$ such that $Q(\mathbf{r})$ is maximal among all $Q(\mathbf{x})$ with $\mathbf{x}$ a unit vector of $\mathcal{V}^3$.

Solution. The matrix of the quadratic form is $A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. The eigenvalues are $\lambda_1 = -1$ (double) and $\lambda_2 = 2$ (simple). The corresponding eigenspaces are

$V_{-1} = L((1,1,0),(1,0,1)))$ and $V_2 = L((-1,1,1))$. Orthonormalizing the basis of V_{-1} and normalizing the generator of V_2 one get an orthonormal basis of eigenvectors $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, with $\mathbf{u}$ parallel to $(1,1,0)$ and $\mathbf{w}$ parallel to $(-1,1,1)$. Letting (x', y', z') the vector of components of (x, y, z) with respect to the basis $\mathcal{B}$, we get the canonical form

$$Q(x, y, z) = -(x')^2 - (y')^2 + 2(z')^2.$$

From this it is easy to answer to the questions (of course, to answer questions (a) and (b) one can make attempts, without using the canonical form).

(a) For example, $\mathbf{u} = (1,1,0)$ and $\mathbf{w} = (-1,1,1)$ are perpendicular vectors such that $Q(\mathbf{u}) < 0$ and $Q(\mathbf{w}) > 0$ (this follows because $\mathbf{u}$ is such that $z' = 0$ and $\mathbf{w}$ is such that $x' = y' = 0$. Moreover they are perpendicular because they are eigenvectors corresponding to different eigenvalues of a symmetric matrix).

(b) For example, for $x' = \sqrt{2}$, $z' = 1$ we get zero. Therefore we get $\mathbf{v} = (1,1,0) + (1/\sqrt{3})(-1,1,1)$ such that $Q(\mathbf{v}) = 0$. (of course, by direct inspection, for this particular quadratic form one immediately sees for example $\mathbf{w} = (0,0,1)$ which is a much simpler).

(c) Here one needs the theory. Indeed we know that the maximal eigenvalue is the maximum value taken by the quadratic form on the unit sphere (that is, the set of all unit vectors). Therefore in our case the maximum of $Q(\mathbf{x})$ for $\mathbf{x}$ unit vector is 2 and, by the canonical form, $Q(\mathbf{r}) = 2$ only for $\mathbf{r} = (1/\sqrt{3})(-1,1,1)$ and $\mathbf{r} = -(1/\sqrt{3})(-1,1,1)$.