L. A. G. 2015. 3rd written exam, 9 - 7, 2015

1. Let $P=(1,1,2)$, $A=(1,-2,2)$ and $B=(4,3,0)$. Let L be the line $\{P+tA \mid t\in\mathbf{R}\}$ and let S be the line $\{P+sB \mid s\in\mathbf{R}\}$.
(a) Write down a pair of points (Q,R) with $Q\in L$ and $R\in S$ such that: $d(P,Q)=d(P,R)$ and the area of the triangle of vertices P,Q,R is 30.
(b) Find *all* pairs of points (Q,R) as in (a) (that is $Q\in L$ and $R\in S$ such that: $d(P,Q)=d(P,R)$ and the area of the triangle of vertices P,Q,R is 30).
(c) Find all planes M containing both the lines L and S and such that $d(M,P)=15$.

(a) Since $L_t=P+tA$, we get $d(L_t,P)=\| tA \|=|t|\ \| A \|=|t|3$. For the same reason $d(S_s,P)=|s|5$. From the condition $d(L_t,P)=d(S_s,P)$ we get $|s|=\frac{3}{5}|t|$. Taking, for example $t>0$ and $s>0$, we get

$$s=\frac{3}{5}t$$

The area, say a, of the triangle is $\frac{1}{2}\ \| tA\times sB \|=\frac{1}{2}|t||s|\ \| A\times B \|=\frac{1}{2}|t||s|\sqrt{221}$. Substituting the previous expression for s we get $a=t^2(\frac{3\sqrt{221}}{5})$. Imposing $a=30$ we get $|t|=\frac{2^{1/2}5}{221^{1/4}}$. Since t is assumed to be positive $t=\frac{2^{1/2}5}{221^{1/4}}$. Consequently $s=\frac{2^{1/2}}{221^{1/4}3}$.
(b) The pairs are 4 (t and s both positive, with negative, one positive and the other negative).
(c) Obviously there are no such planes: if a plane M contains the line L, it contains also the point P, hence $d(P,M)=0$.

2. Let V_3 the linear space of real polynomials of degree ≤ 3. Let $P(x)=x^3+x^2-2x+1$ and $Q(x)=x^3-x^2+2x+1$.
(a) Write down two different bases of V_3 containing both $P(x)$ and $Q(x)$ and such that the remaining elements of the first basis are not scalar multiples of any element of the second basis.
(b) Write down a polynomial $R(x)\in V_3$ which is not a scalar multiple of $P(x)$ or $Q(x)$, such that $\{P(x),Q(x),R(x)\}$ is a linearly dependent set .

(a) For example, $\{1,x,P(x),Q(x)\}$ and $\{1+x,x^2,P(x),Q(x)\}$. Indeed the polynomial are independent if and only if their vectors of coefficients are. In the first case the set of vectors of coefficients are $\{(1,0,0,0),(0,1,0,0),(1,-2,1,1),(1,2,-1,1)\}$ and an easy calculation shows that they are independent. Similarly for the second set.
(b) For example: $R(x)=P(x)+Q(x)=2+2x^3$.

3. For t varying in $\mathbf{R}$ let us consider the matrices

$$A_t=\begin{pmatrix}1&0&0&1\\0&t+2&0&0\\t+1&2&1&0\\0&0&0&2\end{pmatrix}\quad \text{e}\quad B_t=\begin{pmatrix}1&0&0&0\\1&-1&t+2&2\\0&t+2&0&0\\0&0&0&-(t+2)\end{pmatrix}$$

(a) Compute the t's such that A_t is not invertible. Compute the t' s such that B_t is not invertible.
(b) For t varying in $\mathbf{R}$, compute $rk(A_tB_t)$.
(c) For a varying in $\mathbf{R}$, let $C_a = \begin{pmatrix} 1 \\ a+1 \\ a \\ a+1 \end{pmatrix}$. Find for which pairs (t,a) the linear system $(A_tB_t)X = C_a$ has :
- a unique solution.
- no solution.
- infinitely many solutions.

(a) One computes $\det(A_t) = 2(t+2)$. Therefore $A-t$ is not invertible if and only if $t=-2$. Moreover $\det(B_t) = -(t+2)^3$. therefore B_t is not invertible if and only if $t=-2$.
(b) Since the product of two invertible matrices is invertible, we have that, if $t \neq -2$, A_tB_t is invertible, hence $rk(A_tB_t) = 4$. It remains to check the case $t=-2$, where a direct calculations shows that $rk(A_{-2}B_{-2}) = 2$.
(c) By Cramer's theorem, for $t \neq -2$ the solution is unique for each a. It remains to see what happens for $t=-2$. In this case an easy calculation shows that the system $A_{-1}B_{-2}X = C_a$ has a solution if and only if $a=-1$, and in this case they are infinitely many.

4. Let us consider the quadratic form Q defined by $Q(x,y,z,t) = (x+y-z+2t)^2$.
(a) Is Q positive, semi positive or indefinite?
(b) Reduce Q to canonical form. That is: find an orthonormal basis $\mathcal{B}$ of V^4 and scalars a,b,c,d such that $Q(x,y,z,t) = a(x')^2 + b(y')^2 + c(z')^2 + d(t')^2$, where (x',y',z',t') are the components of (x,y,z,t) with respect to the basis $\mathcal{B}$.
(c) Find the maximum and the minimum of Q on the unit sphere of V^4 (recall: the unit sphere of V^4 is the set of unit vectors of V^4).

(a) It is clearly semi-positive, since $Q(x,y,z,t) \geq 0$ for all $(x,y,z,t) \in V^4$, but, $Q(x,y,z,t) -$ 0 for all (x,y,z,t) such that $x+y-z+2t = 0$.

(b) One can find the canonical form by computing the eigenvalues of the associated matrix and orthonormal bases of the corresponding eigenspaces. However, this particular quadratic form is so simple that one can find the canonical form immediately. In fact, knowing what the shape of the canonical form will be (see the textbook, or simply the text of the present exercise), from the fact that for $x+y-z+2t = 0$ (which defines a subspace W of dimension 3) we infer that three of the four scalars a,b,c,d, say b, c and d, must vanish. Moreover the orthonormal basis $\mathcal{B}$ must be such that the last three vectors of $|calB$ form a (orthonormal) basis of W. Hence the first vector has to be a basis of $W^\perp = L((1,1,-1,2))$. whose orthonormal basis is, for example $(1/\sqrt{7})(1,1,-1,2)$. In conclusion: the x' of the exercise is the first component of the vector of components of (x,y,z,t) with respect to $\mathcal{B}$, namely (using what we know about orthonormal bases):

$x' = (1/\sqrt{7})(x, y, z, t)(1, 1, -1, 2) = (1/\sqrt{7})(x + y - z + 2t)$. Form this is follows immediately that the canonical form is $Q(x, y, z, t) = 7(x')^2$. The basis $\mathcal{B}$ will be given by $(1/\sqrt{7})(x, y, z, t)(1, 1, -1, 2)$ plus an orthonormal basis of W.

(c) Maximum: 7. Minimum: 0.