**L. A. G. 2015. first written exam, 6-30, 2015**

**1.** Let us consider the following straight lines: $L = \{(1,1,1) + t(1,2,3)\}$ and $M = \{(3,2,4) + s(1,-1,1)\}$.
(a) Find $P = L \cap M$. (b) Find all points $Q \in M$ such that the area of the triangle of vertices $P$, $(1,1,1)$ and $Q$ is equal to 10.

*Solution.* (a) $P = (2,3,4)$. (b) The area is $\mathcal{A} = \frac{1}{2} \parallel ((1,1,1) - P) \times (Q - P) \parallel$. Since $P$ belongs to the line $M$, it follows that $Q - P = \lambda(1,-1,1)$ for $\lambda \in \mathbf{R}$ (this simplifies the calculation). Therefore $\mathcal{A} = \frac{1}{2}|\lambda| \parallel (-5,-2,3) \parallel$. From this, imposing $\mathcal{A} = 10$ one finds $\lambda$ (there are two values) and therefore $Q = P + \lambda(1,-1,1)$ (there are two solutions).

**2.** A point moves in $V_3$ with position vector $\mathbf{r}(t)$ such that: (a) $\mathbf{v}(t)$ is always perpendicular to $\mathbf{r}(t)$; (b) $\mathbf{a}(t)$ is always parallel to $\mathbf{r}(t)$; (c) $\parallel \mathbf{a}(t) \parallel$ is constant, equal to $K$. Moreover $\mathbf{r}(0) = (1,2,2)$ and $\mathbf{v}(0) = (4,-2,0)$.
Compute the speed $v(t)$, describe the underlying curve (hint: consider te curvature), and compute the constant $K$.

*Solution.* As we know, condition (b) implies that the curve is contained in a plane passing trough the origin, while condition (a) implies that $v(t)$ is constant $\equiv v$. Therefore $\mathbf{a}(t) = v^2\kappa(t)N(t)$ so that condition (c) implies that also the curvature $\kappa(t)$ is constant. Therefore we know that the curve, being a plane curve with constant curvature, is a circle. Since the acceleration is parallel to $\mathbf{r}(t)$, the center is the origin. The radius must be equal to $\parallel \mathbf{r}(0) \parallel = 3$ hence the curvature is $1/3$. The plane is $L((1,2,2),(4,-2,0))$. The speed is $\parallel \mathbf{v}(0) \parallel = \sqrt{20}$. The constant $K$ follows.

**3.** Let $V$ be the space of polynomials of degree $\leq 3$. Say which of the following is an inner product on $V$ and which is not, and why. In the negative case specify which property is not satisfied and show an example where it is not satisfied.
(a) $(P,Q) = |P(0)Q(0)| + |P(1)Q(1)| + |P(-1)Q(-1)|$ ;
(b) $(P,Q) = \int_{-1}^{1} P'(x)Q'(x)dx$
(c) $(P,Q) = P(0)Q(0) + P(1)Q(1) + P(-1)Q(-1)$
(d) $(P,Q) = P(0)Q(0) + P(1)Q(1)$

*Solution.* The answer in NO in all cases.
(a) Example: $P(x) \equiv 1$, $Q(x) \equiv 1$ $\lambda = -1$. Then $(\lambda P, Q) = |-3| = 3$ and $\lambda(P,Q) = -3$. Therefore the property $(\lambda P, Q) = \lambda(P,Q)$ is not always satisfied.
(b) Example: $P(x) \equiv 1$. In this case $(P,P) = 0$ but $P \neq 0$. therefore the positivity property is not always satisfied.
(c)(d) Example: $P(x) = x(x-1)(x+1)$. For both (c) and (d) $(P,P) = 0$ but $P \neq 0$. Therefore the positivity property is not satisfied.

**4.** Let $V$ be the linear space of polynomials of degree $\leq 2$, and let $T : V \to V$ be the following linear transformation: $T(P(x)) = 2P(x) - (x+1)P'(x+1)$. Compute the matrix representing $T$ with respect to the basis $\{1, x, x^2\}$. Compute the null-space and the rank of $T$.

*Solution.* Computing one finds that $T(a + bx + cx^2) = 2a - b - 2c + (b - 4c)x$. Therefore the matrix is $\begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}$ (indeed $T(1) = 2 + 0x + 0x^2$, $T(x) = -1 + 1x + 0x^2$, $T(x^2) = -2 - 4x + 0x^2$).

The null-space is the subspace of polynomials $a + bx + cx^2$ such that $\begin{cases} 2a - b - 2c & =0 \\ b - 4c & =0 \end{cases}$. Solving one gets $b = 4c$ and $a = 3c$. Therefore $N(T) = \{3c + 4c\,x + cx^2 \mid c \in \mathbf{R}\} = L(3 + 4x + x^2)$.

**5.** Let $\mathcal{C}$ be the conic section of cartesian equation: $f(x, y) = x^2 + y^2 + 2xy - 4x + 4y + 4 = 0$. Find the canonical form and find the (x,y)-coordinates of the center/vertex and the equation of the symmetry axes/axis. Draw a rough sketch.

*Solution.* We first reduce to canonical form the quadratic form $x^2 + y^2 + 2xy$. The matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which has 0 and 2 as eigenvalues and respectively $L((1, -1))$ and $L((1, 1))$ as eigenspaces (these will be the $x'$ and $y'$ axes of the canonical form). We take as orthonormal basis $\{\frac{1}{\sqrt{2}}(1, -1), \frac{1}{\sqrt{2}}(1, 1)\}$. The $(x, y)$ and $(x', y')$ coordinates are related by $\begin{cases} x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \\ y = -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \end{cases}$. Then

$$f(x, y) = 2(y')^2 - 4(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y') + 4(-\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y') + 4 = 2(y')^2 - 8\frac{1}{\sqrt{2}}x' + 4$$

Therefore the canonical form is

$$2(y')^2 = \frac{8}{\sqrt{2}}(x' - \frac{1}{\sqrt{2}})$$

The conic section is a parabola. Its vertex is $(x', y') = (\frac{1}{\sqrt{2}}, 0)$. Passing to $(x, y)$-coordinates we get $V = (\frac{1}{2}, \frac{1}{2})$. The symmetry axis is $V + L((1, -1))$.