Alcuni esercizi risolti di geometria di $\mathbf{R}^2$ e $\mathbf{R}^3$

NOTA: questi esercizi sono mutuati dal materiale didattico di un altro corso in cui, purtroppo, si usa la notazione "orizzontale" per i vettori. Trascrivendoli, passate alla notazione verticale. Inoltre mancano esercizi su circonferenze e sfere, che saranno presenti in altri files.

1. Let $A = (1, -1, -1)$ and $B(1, 2, 3)$.
(a) Find a vector C parallel to A and a vector D perpendicular to A such that

$$2A - B = C + D$$

(b) Find all vectors parallel to A whose norm is 9.

Solution. (a) What is required in the ortogonal decomposition of the vectore $2A - B$ in a vector parallel to A and a vector perperdicular to A. Hence

$$C = \left(\frac{(2A - B) \cdot A}{A \cdot A}\right)A = \frac{10}{3}(1, -1, -1)$$

and

$$D = (2A - B) - C = \frac{1}{3}(-7, -2, -5)$$

(b) We look for all $c \in \mathbf{R}$ such that $\| cA \| = 9$. But $\| cA \| = |c| \, \| A \| = |c|\sqrt{3}$. Therefore $|c| = \frac{9}{\sqrt{3}} = 3\sqrt{3}$. Hence there are two solutions $c = 3\sqrt{3}$ and $c = -3\sqrt{3}$. In conclusion, there are two vectors as requested:

$$3\sqrt{3}\,A = 3\sqrt{3}\,(1, -1, -1) \qquad \text{and} \qquad -3\sqrt{3}\,A = -3\sqrt{3}\,(1, -1, -1)$$

2. Let $P = (1, 2, -1)$ and $Q = (0, 1, -2)$. Which of the following points belong to the line containing P and Q?
(a) $P + Q$; (b) $Q + (-1, 2, 1)$; (c) $P + (3, 3, 3)$; (d) $P - 2Q$; (e) $Q + (5, 5, 5)$; (f) $P + (1, 3, 1)$

Solution. We know that a point R belongs to the line containing P and Q if and only if $R - P$ is a scalar multiple of $Q - P$. Equivalently, this can be expressed as: $R - Q$ is a scalar multiple of $Q - P$. Since $Q - P = (-1, -1, -1)$ this is even easier to check (the scalar multiples of that vector are the vectors whose coordinates are equal). Hence: (a) NO; (b) NO; (c) YES; (d) NO; (e) YES; (f) NO.

3. Let $L = \{(1, 2) + t(3, -4)\}$ and $O = (0, 0, 0)$. Compute $d(O, L)$ and the point of L closest to O.

Solution. $d(O, L) = 2$. Closest point: $(1, 2) + (1/5)(3, -4) = (8/5, 6/5)$.
These solutions are obtained by applying directly the formulas. Closest point: writing the line as $\{P + tA\}$, we write $O - P = -P = \big((-P) \cdot A/(A \cdot A)\big)A + C = (1/5)(3, -4) + C$. We have that the closest point is $P + (1/5)A$ and $d(O, L) = \| C \|$.

4. Let $L = \{(-1, -1, -1) + t(1, 1, 0)\}$ and $S = \{(1, 4, -1) + t(0, 1, -1)\}$. Compute $L \cap S$.

Solution. $L \cap S = (1, 4, -1) - 2(0, 1, -1) = (1, 2, 1)$.
This is obtained as follows: a point of the intersection is both of the form $P + tA = Q + sB$. hence (t, s) are

solutions of the system $tA + s(-B) = Q - P$. In our case: $t\begin{pmatrix}1\\1\\0\end{pmatrix} + s\begin{pmatrix}0\\-1\\1\end{pmatrix} = \begin{pmatrix}3\\5\\-2\end{pmatrix}$. The solutions are easily found: $t = 3,\ s = -2$.

5. For each $t \in \mathbf{R}$ and $a \in \mathbf{R}$ consider the system $\begin{cases} 3x + ty + 2z & = 2 \\ 3tx - ty + z & = \mathrm{a} \\ 6x + ty + z & = \mathrm{t} \end{cases}$.
Determine all pairs (t, a) such that the system has a unique solution, no solutions, more than one solution.

Solution. Let $\mathcal{A}(t)$ be the matrix of coefficients of the system. We find that $\det \mathcal{A}(t) = 3t^2 + 12t$. Therefore the system has a unique solution if and only if $t \neq 0, -4$. If $t = 0$: one finds (with gaussian elimination, for example) that there are solutions (necessarily infinitely many) if and only if $a = 4/3$. If $t = -4$ one finds (in the same way) that there are solutions if and only if $a = 16$. In conclusion: MORE THAN ONE (infinitely many) solutions: $(t, a) = (0, 4/3)$ and $(t, a) = (-4, 16)$. NO solutions for $(0, a)$ with $a \neq 4/3$ and for $(-4, a)$ with $a \neq 16$. ONE solution: for every (t, a) with $t \neq 0, -4$.

6. Let $P = (1, 1, -1)$, $Q = (1, 0, 1)$, $R = (2, 2, -1)$. (a) Find the cartesian equation of the plane M containing P, Q and R. (b) Find the cartesian equations of the planes M' parallel to M such that $d(Q, M') = 4$.

Solution. The equation of the plane M is $(X - P) \cdot (Q - P) \times (R - P) = 0$, that is: $-2x + 2y + z = -1$. A parallel plane M' has equation as follows: $-2x + 2y + z = d$, and one has to find the d's such that $d(Q, M') = 4$. We know that $d(Q, M') = |(-2) \cdot 1 + 1 \cdot 1 - d|/3 = |-1 - d|/3$. hence the planes are two, one for $d = -13$ and the other for $d = 11$.

7. Let $P = (1, -1)$ and let $R = L((3, -4)) = \{t(3, -4) \mid t \in \mathbf{R}\}$. Find all lines S parallel to L such that the distance $D(P, S) = 10$.

Solution. A line parallel to R has cartesian equation of the form $4x + 3y = c$, with $c \in \mathbf{R}$. We have to find the c's such that the distance from P is equal to 10. We know that the distance is $\frac{|(4,3)\cdot(1,-1)-c|}{\|(4,3)\|} = \frac{|1-c|}{5}$. We want $\frac{|1-c|}{5} = 10$ which splits into the two possibilities: $1 - c = 50$, and $c - 1 = 50$. therefore there are two lines as required: $S_1 : \ 4x + 3y = -49$ and $S_2 : \ 4x + 3y = 51$.

8. Let $A = (1, 1, -1)$, $B = (5, 2, -4)$ and $C = (2, 2, 3)$. Find all vectors D of the form $xA + yB$ which are orthogonal to C.

Solution. $xA + yB = (x + 5y, x + 2y, -x - 4y)$. Therefore

$$(xA + yB) \cdot C = (x + 5y, x + 2y, -x - 4y) \cdot (2, 2, 3) = 2x + 10y + 2x + 4y - 3x - 12y = x + 2y$$

Therefore $(xA + yB) \cdot C = 0$ if and only if $x = -2y$. This means that the vectors $xA + yB$ which are orthogonal to C are those of the form

$$-2yA + yB = y(-2A + B) = y(3, 0, -2)$$

Now

$$\| \ y(3, 0, -2) = |y| \ \| \ (3, 0, -2) \ \| = |y|\sqrt{13}.$$

Therefore the norm is 10 if and only if $|y|\sqrt{13} = 10$, that is $y = 10/\sqrt{13}$ and $y = -10/\sqrt{13}$. The final answer is: there are two vectors as required:, namely

$$D_1 = \frac{10}{\sqrt{13}}(3, 0, -2) \qquad \text{and} \qquad D_2 = \frac{-10}{\sqrt{13}}(3, 0, -2).$$

9. In $\mathbf{R}^3$, let us consider the plane $M = \{(0, 0, 1) + t(1, 0, 1) + s(1, -1, 0)\}$, and the line $L = \{t(1, 1, 1)\}$. (a) Find all points in L such that their distance from M is equal to $\sqrt{3}$. (b) For each such point P find the cartesian equation of the plane parallel to M containing P.

Solution. (a) A normal vector to the plane is $(1, 0, 1) \times (1, -1, 0) = (1, 1, -1)$. A point of the line is of the form $t(1, 1, 1) = (t, t, t)$. Its distance form the plane is

$$\frac{|((t, t, t) - (0, 0, 1)) \cdot (1, 1, -1)|}{\| (1, 1, -1) \|} = \frac{|t + 1|}{\sqrt{3}}|$$

Hence $|t + 1|\sqrt{3} = \sqrt{3}$, that is $|t + 1| = 3$, which has the two solutions $t = 2$ and $t = -4$. The requested points are $(2, 2, 2)$ and $(-4, -4, -4)$.
(b) A cartesian equation of a plane parallel to M is of the form $x + y - z = d$. Replacing $(2, 2, 2)$ we find $d = 2$ and replacing $(-4, -4, -4)$ we find $d = -4$.

10. Consider the lines in $\mathbf{R}^3$: $L = \{(2, 4, -2) + t(1, 1, 0)\}$ and $M = \{(-3, 7, -4) + t(2, -2, 1)\}$.
(a) Show that the intersection of L and M is a point and find it. Let us denote it P.
(b) Find a point $Q \in L$ and a point $R \in M$ such that $\| Q - P \| = \| R - P \|$ and the area of the triangle with vertices P, Q, R is 2.

Solution. (a) The intersection point (if any) is a point P such that there are a $t \in \mathbf{R}$ a $s \in \mathbf{R}$ such that $P = (2, 4, -2) + t(1, 1, 0) = (-3, 7, -4) + s(2, -2, 1)$. Therefore $t(1, 1, 0) - s(2, -2, 1) = (-5, 3, -2)$. One finds easily, for example, that $-s = 2$. Hence $s = -2$ so the intersection point is $P = (-3, 7, -4) - 2(2, -2, 1) = (1, 3, -2)$.
(b) We can write $L = \{(1, 3, -2) + s\frac{1}{\sqrt{2}}(1, 1, 0)\}$ and $M = \{(1, 3, -2) + s\frac{1}{3}(2, -2, 1)\}$. Therefore $Q - P = s(1/\sqrt{2}(1, 1, 0))$ and $\| Q - P \| = |s|$. Moreover $R - P = \lambda(1/3(2, -2, 1)$ and $\| R - P \| = |\lambda|$. Since we want $\| Q - P \| = \| R - P \|$, we can take $s = \lambda$. Moreover the area of the triangle with vertices P, Q, R is $1/2 \| (Q - P) \times (R - P) \| = (1/2)(\sqrt{18}/3\sqrt{2})s^2 = s^2/2$. Since the area has to be equal to 4, we can take $s = 2$ (or $s = -2$). Therefore $Q = (1, 3, -2) + 2(1/\sqrt{2})(1, 1, 0)$ and $R = (1, 3, -2) + 2/3(2, -2, 1)$ are points satisfying the requests of (b).

11. In $\mathbf{R}^3$, let us consider the two straight lines $L = \{(-2, -5, 4) + t(1, 1, -1)\}$ and $R = \{(0, 1, 3) + t(-1, 3, 2)\}$.
(a) Compute the intersection $L \cap R$.
(b) Is there a plane containing both L and R? if the answer is yes, compute its cartesian equation.

Solution. (a) A point P lies in the intersection if and only if there are scalars t_0 and s_0 such that $(-2, -5, 4) + t_0(1, 1, -1) = (0, 1, 3) + s_0(-1, 3, 2) = P$. Therefore $t_0(1, 1, -1) - s_0(-1, 3, 2) = (2, 6, -1)$. Solving the system we find $-s_0 = 1$, hence $s_0 = -1$. Therefore $P = (0, 1, 3) - (-1, 3, 2) = (1, -2, 1)$.
(b) The plane is $\{(1, -2, 1) + t(1, 1, -1) + s(-1, 3, 2)\}$. Cartesian equation: $5x - y + 4z = 11$.

12. Let $A = (1, 1, -1)$, $B = (-1, 1, 1)$ and $C = (1, 2, 1)$. Find two vectors D and E such that: $C = D + E$, D is orthogonal to both A and B, and E is a linear combination of A and B. Is there a unique solution?

Solution. We know that C can be written in a unique way as the sum of two vectors D and E such that D is parallel to $A \times B$ and E is orthogonal to $A \times B$ (hence E belongs to $L(A, B)$). These are the requested vectors D and E, and we know that they are the unique vectors having such properties. We compute: $A \times B = (2,0,2)$. To simplify the computation we can take $(1,0,1)$ rather than $(2,0,2)$.

$$D = \frac{(1,2,1)\cdot(1,0,1)}{2}(1,0,1) = (1,0,1), \qquad E = (1,2,1) - D = (0,2,0)$$

Note $E = (0,2,0) = A + B$. Now $C = D + E$.

13. Let $A = (1,1,1)$, $B = (1,-1,-1)$ and, for t varying in $\mathbf{R}$, $C(t) = (1,t,2)$. Find the values of t such that one of the three vectors A, B, $C(t)$ can be expressed as a linear combination of the remaining two, and, for each such value of t, write explicitly one such expression.

Solution. A vector of A, B and $C(t)$ can be expressed as a linear combination of the remaining two if and only if $\{A, B, C(t)\}$ is a set of linearly dependent vectors. This happens if and only if the triple product $A\cdot(B \times C(t))$ is zero. The triple product is the determinant $\det\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & -1\\ 1 & t & 2\end{pmatrix} = 2t-4$, which is zero only for $t = 2$. In this case, solving the system $xA + yB + zC(2) = O$ one easily finds the solutions $(-3y, y, 2y)$, $y \in \mathbf{R}$. For $y = 1$ one gets $(-3,1,2)$ which means that $-3A + B + 2C(2) = O$. Hence $B = 3A - 2C(2)$ (check!).

14. Let us consider the two lines in $\mathbf{R}^2$: $L = \{(1,-1) + t(1,2)\}$ and $S : x + y = -3$. How many are the points $P \in \mathbf{R}^2$ such that $\begin{cases} d(P,L) = \sqrt{5} \\ d(P,S) = \sqrt{2} \end{cases}$? Find explicitly two of them.

Solution. The points are four. In fact the locus of points P whose distance from L (respectively M) is a fixed number, say d, is the union of two lines, parallel to L (resp. parallel to M))(one in each half-plane). Hence the intersection of the two loci is four points. In our specific case, let $P = (x,y)$. We have that

$$d((x,y),L) = \frac{|(x-1,y+1)\cdot(2,-1)|}{\sqrt{5}} = \frac{|2x-y-3|}{\sqrt{5}}$$

Moreover

$$d((x,y),S) = \frac{|(x,y)\cdot(1,1)+3|}{\sqrt{2}} = \frac{|x+y+3|}{\sqrt{2}}$$

Hence we are looking for points (x,y) such that

$$\begin{cases} \frac{|2x-y-3|}{\sqrt{5}} = \sqrt{5} \\ \frac{|x+y+3|}{\sqrt{2}} = \sqrt{2} \end{cases}$$

A first case is, for example,

$$\begin{cases} 2x - y - 3 = 5 \\ x + y + 3 = 2 \end{cases}$$

hence $P = (7/3, -10/3)$. A second case is, for example,

$$\begin{cases} -2x + y + 3 = 5 \\ x + y + 3 = 2 \end{cases}$$

hence $P = (-1,0)$.

15. Let $A = (1,0,-1)$, $B = (1,2,1)$. Moreover let L be the line $L = (1,0,0) + t(1,-1,1)$, $t \in \mathbf{R}$.
Find (if any) all points $Q \in L$ such that the area of the triangle whose vertices are A, B and Q is equal to 4.

Solution. The area of a triangle whose vertices are the points A,B, and Q is $\frac{1}{2} \parallel (B-A)\times(Q-A) \parallel$. Since a point of L is of the form $(1,0,0)+t(1,-1,1)$ we get

$$\frac{1}{2} \parallel (0,2,2)\times(t,-t,t+1) \parallel = \frac{1}{2} \parallel (4t+2,2t,-2t) \parallel = \sqrt{6t^2+4t+1}$$

Hence we are looking for the t's such that $\sqrt{6t^2+4t=1}=4$, hence $t_1 = \frac{-2+\sqrt{94}}{6}$ and $t_2 = \frac{-2-\sqrt{94}}{6}$. The required points are $Q_1 = (1,0,0)+t_1(1,-1,1)$ and $Q_2 = (1,0,0)+t_2(1,-1,1)$

16. Let L be the line $\{(1,0,1)+t(1,0,2)\}$. Moreover let M be the plane passing through the points $(0,1,0)$, $(1,2,-1)$, $(1,1,1)$. Find (if any) all points P of L such that $d(P,M)=1$.

Solution. $M=(0,1,0)+s(1,1,-1)+t(1,0,1)$. The cartesian equation of M is

$$((x,y,z)-(0,1,0))\cdot((1,1,-1)\times(1,0,1))=0,$$

hence $M: x-2(y-1)-z=0$, or $M: x-2y-z=-2$. A normal vector is $N=(1,-2,-1)$ and $Q\cdot N=-2$ for each $Q\in M$.
Let $P=(x,y,z)$. We have that $d(P,M)=\frac{|P-Q|}{\|N\|}$, where Q is any point of M. Therefore

$$d(P,M)=\frac{|P\cdot N+2|}{\sqrt{6}}=\frac{|x-2y-z-2|}{\sqrt{6}}$$

If $P\in L$, we have that $P=(x,y,z)=(1+t,0,1+2t)$. Substituting in the formula for the distance we get $d(P,M)=\frac{|-t-2|}{\sqrt{6}}$. hence we have to solve the equation: $\frac{|-t-2|}{\sqrt{6}}=1$. the solutions are $-2+\sqrt{6}$ and $\sqrt{6}-2$. Plugging these two values of t in the parametric equation of L we get the two points in L whose distance from M is equal to 1.

17. Let A, B, C be three vectors in $\mathbf{R}^3$. Prove or disprove the following assertions:
(a) If A, B and C are linearly independent then the vectors $A+2B$, $A+B-C$, $A+B$ are linearly independent.
(b) The vectors $A+2B$, $A+B-C$, $A+B$ can be linearly independent even if A, B and C are linearly dependent
(c) The vectors $A+2B$, $A+B-C$, $-A+2C$ are always linearly dependent, regardless of the linear dependence or independence of A, B, C.

Solution. (a) is correct. Let $x,y,z\in\mathbf{R}$ such that $O=x(A+2B)+y(A+B-C)+z(A+B)=(x+y+z)A+(2x+y+z)B-yC$. Since A, B and C are independent, this means that $\begin{cases} x+y+z=0 \\ 2x+y+z=0 \\ y=0 \end{cases}$.
It follows easily that $x=y=z=0$.
(b) is false, because the fact that A, B and C are dependent means that they are contained in a plane trough the origin. Hence also all linear combinations of A, B and C are contained in that plane.

18. Let $P=(1,2,1)$ and let R be the line $\{(1,5,-1)+t(1,1,-1)\}$. Compute the distance between P and L and find the point of L which is nearest to P.

Solution. Let $\mathbf{u}=(1,2,1)-(1,5,-1)=(0,-3,2)$. We have that the projection of $\mathbf{u}$ along $L((1,1,-1)$ is $\mathrm{pr}(\mathbf{u})=-5/3(1,1,-1)$. Therefore the nearest point is

$$H=(1,5,-1)-5/3(1,1,-1)$$

The distance is

$$\parallel \mathbf{u}-\mathrm{pr}(\mathbf{u}) \parallel = \parallel (0,-3,2)+5/3(1,1,-1) \parallel = \parallel 1/3(5,-4,-1) \parallel = \sqrt{42}/3$$

19. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three mutually orthogonal vectors of $\mathbf{R}_3$ with $\| \mathbf{v}_1 \| = 4$, $\| \mathbf{v}_2 \| = \| \mathbf{v}_3 \| = 2$. Let $\mathbf{v} = (1/2)\mathbf{v}_1 + (1/2)\mathbf{v}_2 - \mathbf{v}_3$. For $i = 1, 2, 3$ let θ_i be the angle between $\mathbf{v}$ and $\mathbf{v}_i$.
(a) Find θ_i for $i = 1, 2, 3$.
(b) Let $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$. Write $\mathbf{w}$ as the sum of a vector parallel to $\mathbf{v}$ and a vector perpendicular to $\mathbf{v}$.

Solution. (a) Since the three vectors are orthogonal, we have that

$$\| \mathbf{v} \|^2 = \| (1/2)\mathbf{v}_1 \|^2 + \| (1/2)\mathbf{v}_2 \|^2 + \| -\mathbf{v}_3 \|^2 = (1/4) \| \mathbf{v}_1 \|^2 + (1/4) \| \mathbf{v}_2 \|^2 + \| \mathbf{v}_3 \|^2 = 9$$

Hence $\| \mathbf{v} \| = 3$. Therefore $\cos\theta_1 = \frac{\mathbf{v}_1 \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}_1\|} = \frac{(1/2)\mathbf{v}_1 \cdot \mathbf{v}_1}{\|\mathbf{v}\| \|\mathbf{v}_1\|} = 2/3$. (here we used that $\mathbf{v_1} \cdot \mathbf{v}_2 = \mathbf{v_1} \cdot \mathbf{v}_3 = 0$. Similarly, one computes $\cos\theta_2 = 1/3$ and $\cos\theta_3 = -2/3$.

20. Let L be the line of $\mathbf{R}^3$: $L = L\big((1,1,2)\big)$ and let π be the plane of $\mathbf{R}^3$: $\pi = L\big((1,2,0),(0,1,1)\big)$. Find the cartesian equations of all planes of $\mathcal{V}_3$ which are parallel to L, perpendicular to π, and such that their distance from the point $P = (1,3,1)$ is equal to $\sqrt{3}$. *(Recall that two planes are said to be perpendicular if and only if and only their normal vectors are perpendicular).*

Solution. A normal vector to π is $(1,2,0) \times (0,1,1) = (2,-1,1)$. A normal vector to a plane parallel to L has to be perpendicular to $(1,1,2)$. A normal vector to a plane perpendicular to π has to be perpendicular to $(2,-1,1)$. Hence a normal vector to a plane which is both parallel to L and perpendicular to π has to be parallel to $(1,1,2) \times (2,-1,1) = (3,-3,3)$. Hence the cartesian equation of the planes we are looking for is of the form

$$x - y + z = d.$$

Denoting Q a point of our plane (any point), the distance of such a plane from the point $P = (1,3,1)$ is

$$\frac{|(P-Q) \cdot N)|}{\| N \|} = \frac{|P \cdot N - d|}{\| N \|} = \frac{|-1-d|}{\sqrt{3}}$$

Therefore we have to impose

$$\frac{|-1-d|}{\sqrt{3}} = \sqrt{3}$$

yielding $d = -4$ and $d = 2$. In conclusion the required planes are two, having as cartesian equations

$$x - y + z = -4 \qquad \text{and} \qquad x - y + z = 2$$

21. Let $\mathbf{v}_1 = (1,2,2)$ and $\mathbf{v}_2 = (1,0,1)$.
(a) Find three mutually orthogonal vectors of $\mathbf{R}^3$, say $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ such that $L(\mathbf{v}_1) = L(\mathbf{w}_1)$ and $Span(\mathbf{v}_1, \mathbf{v_2}) = Span(\mathbf{w}_1, \mathbf{w}_2)$.
(b) Find three mutually orthogonal vectors of $\mathbf{R}^3$, say $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $Span(\mathbf{v}_1, \mathbf{v_2}) = Span(\mathbf{u}_1, \mathbf{u}_2)$, and the angle between $\mathbf{v}_1$ and $\mathbf{u}_1$ is equal to $\pi/4$.

Solution. (a) We take $\mathbf{w}_1 = \mathbf{v}_1$. Then we find $\mathbf{w}_2$ using the formula

$$\mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{1}{3}(2,-2,1)$$

Therefore we can take $\mathbf{w}_2 = (2,-2,1)$. Then we can take $\mathbf{w}_3 = \mathbf{w}_1 \times \mathbf{w}_2 = (6,3,-6)$.
(b) Clearly we can take $\mathbf{u}_3 = \mathbf{w}_3 = (6,3,-6)$. A convenient way to find $\mathbf{u}_1$ and $\mathbf{u}_2$ is as follows: we first normalize $\mathbf{v}_1 = \mathbf{w}_1$ and $\mathbf{w}_2$, thus obtaining two orthogonal unit vectors, say $\mathbf{w}_1'$ and $\mathbf{w}_2'$, parallel respectively to $\mathbf{v}_1$ and $\mathbf{w}_2$. At this point, two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ as required are, for example, $\mathbf{u}_1 = (1/\sqrt{2})\mathbf{w}_1' + (1/\sqrt{2})\mathbf{w}_2'$ and $\mathbf{u}_2 = (1/\sqrt{2})\mathbf{w}_1' - (1/\sqrt{2})\mathbf{w}_2'$ (here $(1/\sqrt{2}) = \cos(\pi/4) = \sin(\pi/4)$). Hence

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1,0,1) \qquad \text{and} \qquad \mathbf{u}_2 = \frac{1}{3\sqrt{2}}(-1,4,1)$$

22. Let us consider the planes $M = \{(1,0,2) + t(1,2,1) + s(0,1,-2)\}$ and $N = \{(1,3,0) + t(1,1,1) + s(1,0,-1)\}$. Describe and find equations for the set of points $P \in \mathcal{V}_3$ such that $d(P, M) = d(P, N)$.

Solution. Letting $P = (x, y, z)$, we are interested in the set defined by the equation

$$d((x,y,z), M) = d((x,y,z), N) \tag{1}$$

To find an explicit expression for the left and right hand side, we compute the cartesian equations of the two planes. They turn out to be

$$M \,:\, 5x - 2y - z = 4 \qquad \text{and} \qquad N:\, x - 2y + z = 1$$

Therefore

$$d((x,y,z), M) = \frac{|5x - 2y - z - 4|}{\sqrt{6}} \qquad \text{and} \qquad \frac{d((x,y,z), N) = |x - 2y + z - 1|}{\sqrt{30}}$$

Therefore (1) becomes

$$\frac{|5x - 2y - z - 4|}{\sqrt{6}} = \frac{|x - 2y + z - 1|}{\sqrt{30}}$$

which is equivalent to

$$\frac{5x - 2y - z - 4}{\sqrt{6}} = \frac{x - 2y + z - 1}{\sqrt{30}} \qquad \text{or} \qquad \frac{5x - 2y - z - 4}{\sqrt{6}} = -\frac{x - 2y + z - 1}{\sqrt{30}}$$

which is the equation of the union of two planes.