GEOMETRIA 2015
ALCUNI ESERCIZI DI GEOMETRIA IN $\mathbf{R}^2$ e $\mathbf{R}^3$

NOTA: questi esercizi vengono dal materiale didattico diun altro corso dove si usa la notazione "orizzontale" per i vettori. Voi, studenti del corso GEOMETRIA 2015, tras riveteli e risolveteli con la notazione verticale.

1 . Let $A = (1, 1, -1)$, $B = (1, 2, 0)$, $C = (1, 1, 1)$. (a) Are A, B, C linearly independent? (b) Find scalars $X, y, z \in \mathbf{R}$ such that $(0, 1, 1)xA + yB + zC$.

2. Let $A = (1, 1, -1)$, $B = (1, 2, 0)$, $C = (-2, -5, -1)$. (a) Are A, B, C linearly independent? (b) If the answer to (a) is in the negative, express a A,B or C as a linear combination of the remaining two vectors.

3. For t varying in $\mathbf{R}$, let $A_t = (t + 1, 1, 1)$, $B_t = (1, t + 2, 1)$, $C_t = (1, 1, t + 2)$.
(a) Find all $t \in \mathbf{R}$ such that A_t, B_t, C_t are linearly independent.
(b) Compute, for each $h \in \mathbf{R}$, the solutions of the system (depending on h): $\begin{cases} hx + y + z = 0 \\ x + hy + z = 0 \\ x + y + hz = 0 \end{cases}$.

4. For t varying in $\mathbf{R}$, let $A(t) = (t + 1, t + 2, 2t)$, $B(t) = (1, t^2 + 1, 1)$, $C(t) = (1, 2, 1)$. (a) Find all t such that $A(t)$, $B(t)$ and $C(t)$ do not form a basis of $\mathcal{V}_3$. (b) For all $t, s \in$ let us consider the system

$$xA(t) + yB(t) + zC(t) = (s, 2s, 1).$$

For which values of t and s there is no solution? For which values of t and s there is more than one solution?

5. For t, a varying in $\mathbf{R}$, let us consider the system $\begin{cases} x + y + z = 1 \\ 2x + ty + z = -1 \\ 6x + 7y + 3z = a \end{cases}$. Find for which values of t and a there is a unique solution, no solution, infinitely many solutions.

6. Let $A = (3, 4)$. Describe the set of all vectors $X \in \mathbf{R}^2$ such that the angle between A and X is $\pi/4$.

7. (a) Let us consider the lines of $\mathbf{R}^2$: $L : 2x - y + 1 = 0$ and $R = \{(3, 1) + t(4, 3)\}$. Describe and find cartesian equations for the following subset of $\mathbf{R}^2$:

$$\{X \in \mathbf{R}^2 \mid d(X, L) = d(X, R)\}.$$

(b) Let $L = \{(1, 2) + t(1, 3)\}$ and $R = \{(-2, 5) + t(2, 6)\}$. Describe and find cartesian equations for the following subset of $\mathbf{R}^2$:

$$\{X \in \mathbf{R}^2 \mid d(X, L) = d(X, R)\}.$$

(c) Let L and R be two non-parallel lines in $\mathbf{R}^2$. Describe the subset of $\mathbf{R}^2$ whose elements are points $X \in \mathcal{V}_2$ such that $d(X, L) = d(X, R)$. What happens if L and R are parallel?
(d) For L and R as in (a), describe and find cartesian equations for the following subset of $\mathbf{R}^2$:

$$\{X \in \mathbf{R}^2 \mid d(X, L) = 2d(X, R)\}.$$

8. (a) Find the intersection of the lines $\{(1, 1) + t(1, 2)\}$ and $\{(1, 1) + t(2, 1)\}$. (b) Same question with the two lines $\{(2, -1, 2) + t(1, 1, -1)\}$ and $\{(2, 0, 0) + t(1, 0, 1)\}$.

9. Let us consider the lines of $\mathbf{R}^2$: $L = (1, -1) + t(1, 1)$ and $R : x + 2y = 1$. How many are the points $X \in \mathbf{R}^2$ such that $\begin{cases} d(X, L) = \sqrt{2} \\ d(X, R) = \sqrt{3} \end{cases}$? Find one of them.

10. Let $A = (1, -2)$ and $P = (1, -1)$. Find both parametric and cartesian equations of the lines passing through P which are parallel to a vector B whose angle with A is $\theta = \pi/4$.

11. Find the line containing $(1,-1,1)$, parallel to the plane $M : x - 3y = 2$, and perpendicular to $A = (2,1,1)$.

12. Let $M = \{(1,2,-1) + s(1,2,2) + t(1,-1,0)\}$ and $U = \{(0,2,3) + s(1,1,1) + t(-1,2,3)\}$. Find a parametric equation of the line $M \cap U$.

13. For each of the following pairs of lines L and S in $\mathbf{R}^3$ compute the intersection of L and R (note: the intersection can be empty) and answer to the following question: is there a plane containing both L and R?
(a) $L = \{(1,0,-1) + t(1,1,1)\}$ $\quad R = \{(2,0,2) + t(1,2-1)\}$
(b) $L = \{(1,0,0) + t(1,1,1)\}$ $\quad R = \{(2,0,2) + t(1,2-1)\}$
(c) $L = \{(1,0,0) + t(1,1,1)\}$ $\quad R = \{(2,0,2) + t(1,1,1)\}$

14. Let $L = \{(1,1) + t(1,-2)\}$ and let $Q = (1,0)$. (a) Compute $d(Q,L)$ and find the point $\bar{P}$ of L which is nearest to Q. (b) Find all points $S \in L$ such that the triangle whose vertices are $Q, \bar{P}$ and S is isosceles (this means that two edges have same length). Compute the area of these triangles.

15. Let $A = (1,-2,2)$ and $B = (1,2,0)$. Find a vector $C \in Span(A,B)$ such that the angle between A and C is $\pi/4$ and the area of the parallelogram determined by A and C is 1.

16. Let $A = (1,1,1)$ and $B = (1,-1,1)$. (a) Find a three mutually perpendicular vactors C, D, E of $\mathbf{R}^3$ such that C is parallel to A and D belongs to $Span(A,B)$.

17. Let L be the line $\{(1,0,1) + t(1,0,2)\}$, and let M be the plane $\{(0,1,0) + t(1,1,-1) + s(1,0,1)\}$. Moreover let $Q = (1,0,0)$.
(a) Find all points X belonging to L such that $d(X,M) = 1$.
(b) Find all points X belonging to L such that $d(X,Q) = 1$.

18 . Let L be the intersection of the two planes $M_1 : x - y + z = 1$ and $M_2 : y - 2z = -1$. Let moreover Π be the plane $\{(1,2,1) + t(1,-1,-) + s(-1,0,2)\}$. Find (if any) a plane containing L and perpendicular to Π. Is such plane unique?

19. Let $A = (1,2,-2)$ and $P = (1,-5,-2)$. Find two planes orthogonal to A such that their distance from Q is 6.

20. Let $M = \{(3,1,1) + t(-2,1,0) + s(2,1,2)\}$. Find the plane parallel to M such that their distance from $(1,1,1)$ is 2. For each of such planes, find the nearest point to $(1,1,1)$.

21. Let $M_1 : x - 3y + 2z = 0$ and $M_2 : 2x - 3y + z = 3$. Let moreover $A = (1,0,-1)$.
(a) Find the plane M containing the line $L = M_1 \cap M_2$ and parallel to A.
(b) Let $Q = (0,-1,1)$. Find the line R which is perpendicular to M and contains Q.
(c) Find $R \cap M$.

22. let $P = (0,-1,0)$, $Q = (1,0,-1)$, $R = (1,-1,1)$.
(a) Find the area of the triangle whose vertices are P, Q and R.
(b) Let $S = (1,-1,-2)$. Compute the volume of the parallepiped determined by the vectors $Q - P$, $R - P$ and $S - P$.
(c) Find the points X, collinear with P and S such that the volume of the parallepiped determined by the vectors $Q - P$, $R - P$ and $X - P$ is 2.

23. For t varying in $\mathbf{R}$, let $A_t = (t+2,1,1)$, $B_t = (1,t+2,1)$, $C_t = (1,1,t+2)$.
(a) Find all $t \in \mathbf{R}$ such that A_t, B_t, C_t are linearly independent. For such t compute (as a function of t) the volume of the parallelepiped determined by A_t, B_t, C_t. In particular, compute such volume for $t = -2$.
(b) Compute, for each $h \in \mathbf{R}$, the solutions of the system (depending on h): $\begin{cases} hx + y + z = 0 \\ x + hy + z = 0 \\ x + y + hz = 0 \end{cases}$.

24. For t, a varying in $\mathbf{R}$, let us consider the system $\begin{cases} x + y + z = 1 \\ 2x + ty + z = -1 \\ 6x + 7y + 3z = a \end{cases}$. Find for which values of t and a there is a unique solution, no solution, infinitely many solutions.

25. Let $A = (2, -1, 2)$ and $B = (1, 0, -1)$ (note that $A \cdot B = 0$). Describe and find parametric equations for the subset of vectors $X \in \mathcal{V}_3$ which are orthogonal to B and such that the angle between A and X is $\pi/4$.