L. A. G. 8th test, June 25 2014

Ex. 1 Let $W = L((1,0,-1,1),(1,1,0,-2))$

Let $R_W : V_4 \to V_4$ be the reflection of V_4 with respect to W.

Let $P_{W^\perp} : V_4 \to V_4$ be the projection of V_4 onto $W^\perp$.

Let $T : V_4 \to V_4$ denote the composed linear transformation:

$$T = P_{W^\perp} \circ R_W$$

(in practice $T(\underline{v}) = P_{W^\perp}(R_W(\underline{v}))$).

(a) Describe $N(T)$ and $T(V_4)$.

(b) Find a basis of V_4 whose elements are eigenvectors of T

(c) Compute $m_{\mathcal{E}}^{\mathcal{E}}(T)$ (where $\mathcal{E}$ denotes the standard basis of V_4).

Ex. 2 Let $T : V_2 \to V_2$ be the linear transformation such that $T((1,2)) = (3,6)$ and $T((2,1)) = (3,6)$.

(a) Find $N(T)$

(b) Compute $m_{\mathcal{E}}^{\mathcal{E}}(T)$, where $\mathcal{E}$ is the standard basis of V_2.

(c) Find a basis $\mathcal{B}$ of V_2 such that $m_{\mathcal{B}}^{\mathcal{B}}(T)$ is a diagonal matrix.

SOLUTION.

Ex. 1 Idea: Let $\{\underline{w}_1, \underline{w}_2\}$ be a basis of W and let $\{\underline{w}_3, \underline{w}_4\}$ be a basis of $W^\perp$. Then $\mathcal{B} = \{\underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4\}$ is a basis of V_4 and we have:

$$T(\underline{w}_1) = P_{W^\perp}(R_W(\underline{w}_1)) = P_{W^\perp}(\underline{w}_1) = \underline{0}$$

$$T(\underline{w}_2) = P_{W^\perp}(R_W(\underline{w}_2)) = P_{W^\perp}(\underline{w}_2) = \underline{0}$$

$$T(\underline{w}_3) = P_{W^\perp}(R_W(\underline{w}_3)) = P_{W^\perp}(-\underline{w}_3) = -\underline{w}_3$$

$$T(\underline{w}_4) = P_{W^\perp}(R_W(\underline{w}_4)) = P_{W^\perp}(-\underline{w}_4) = -\underline{w}_4$$

Therefore $\mathcal{B}$ is a basis of eigenvectors of T and $0, -1$ are the eigenvalues.
Specifically:

(a) From the above we have that $N(T) \supseteq W$ and $T(V_4) \supseteq W^\perp$.
But, by the nullity + rank Theorem,

$$\dim(N(T)) + \dim(T(V_4)) = 4.$$

It follows that $N(T) = W$ and $T(V_4) = W^\perp$.

(b) we can take as $\underline{w}_1 = (1,0,-1,1)$ and $\underline{w}_2 = (1,1,0,-2)$.
To find a basis of $W^\perp$, recall that $W^\perp$ is the space of solutions of the system

$$\begin{cases} x \quad\;\; -z+t=0 \\ x+y \quad -2t=0 \end{cases} \qquad \begin{cases} x = z-t \\ y = -x+2t = -z+3t \end{cases}$$

Therefore
$$W^\perp = \{ \underset{\underset{z(1,-1,1,0)+t(-1,3,0,1)}{\parallel}}{(z-t, -z+3t, z, t)} \mid z, t \in \mathbb{R} \} = \mathcal{L}((1,-1,1,0), (-1,3,0,1))$$

Therefore we can take $\underline{w}_3 = (1,-1,1,0)$ $\underline{w}_4 = (-1,3,0,1)$.

Hence $\mathcal{B} = \{\underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4\}$ is a basis of V_4 whose elements are eigenvectors of T (of eigenvalues $0, 0, -1, -1$ respectively).

(c) let $C = m_{\mathcal{E}}^{\mathcal{B}}(id) = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 3 \\ -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}$.

By the diagram

$$\begin{tikzcd} \overset{\mathcal{E}}{V_4} \arrow[r, "T"] \arrow[d, "id"'] & V_4^{\mathcal{E}} \\ \underset{\mathcal{B}}{V_4} \arrow[r, "T"] & \underset{\mathcal{B}}{V_4} \arrow[u, "id"'] \end{tikzcd}$$

we have:

$$m_{\mathcal{E}}^{\mathcal{E}}(T) = C \operatorname{diag}(0,0,-1,-1) C^{-1}$$

Ex. 2 (a) It is clear that $T(V_2) = L((3,6))$.

Therefore $\dim(N(T)) = 2 - 1 = 1$.

But $(1,2) - (2,1) = (-1,1)$ belongs to $N(T)$

(indeed $T((1,2)) - T((2,1)) = T((1,2)-(2,1)) = (3,6)-(3,6) = (0,0)$)

therefore $\boxed{N(T) = L((-1,1))}$.

(b) By the diagram

$$\begin{tikzcd} \overset{\mathcal{E}}{V_2} \arrow[r, "T"] \arrow[d, "id"'] & \overset{\mathcal{E}}{V_2} \\ \underset{\mathcal{B}}{V_2} \arrow[ur, "T"'] & \end{tikzcd}$$

we have that $m^{\mathcal{E}}_{\mathcal{E}}(T) = m^{\mathcal{B}}_{\mathcal{E}}(T)\, m^{\mathcal{E}}_{\mathcal{B}}(id)$

where $\mathcal{B}$ is any basis of V_2.

Taking as $\mathcal{B} = \{(1,2),(2,1)\}$ we have that

$m^{\mathcal{B}}_{\mathcal{E}}(T) = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix}$. On the other hand $m^{\mathcal{E}}_{\mathcal{B}}(id) = \left(m^{\mathcal{B}}_{\mathcal{E}}(id)\right)^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1}$

For example, using determinants, we get $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

Therefore $m^{\mathcal{E}}_{\mathcal{E}}(T) = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix}\left(-\frac{1}{3}\right)\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -3 & -3 \\ -6 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

(c) The basis $\mathcal{B}$ is easily found: since $T((1,2)) = (3,6) = 3(1,2)$ we have that $(1,2)$ is an eigenvector of eigenvalue $\lambda_1 = 3$. Since $T((-1,1)) = (0,0) = 0(-1,1)$ we have that $(-1,1)$ is eigenvector of eigenvalue $\lambda_2 = 0$. It follows that, with respect to the basis $\mathcal{C} = \{(1,2),(-1,1)\}$ $m^{\mathcal{C}}_{\mathcal{C}}(T) = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$.

Otherwise: take as $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ $P_A(\lambda) = \det\begin{pmatrix} \lambda-1 & -1 \\ -2 & \lambda-2 \end{pmatrix} =$

$= (\lambda-1)(\lambda-2) - 2 = \lambda^2 - 3\lambda = \lambda(\lambda-3)$.

Therefore the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 0$. Computing the eigenspaces one finds $E_3 = L((1,2))$ and $E_0 = L((-1,1))$.