EXERCISES 21-6-2014

Ex. 0.1. *Let* $v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ *and* $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ *Find (if any) all matrices* $A \in \mathcal{M}_{3,3}(\mathbb{R})$ *such that:* $Tr(M) = 4$, $M\,v = 2v$, $M\,w = -w$ *and* A *has an eigenvector orthogonal both to* v *and* w.

Ex. 0.2. *(a) Let* $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *Find (if any) all symmetric matrices* $A \in \mathcal{M}_{2,2}(\mathbb{R})$ *such that* -3 *is an eigenvalue of* A *and* $Av = 2v$.
(b) Let $w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. *Is there a symmetric matrix* $B \in \mathcal{M}_{2,2}(\mathbb{R})$ *such that* $Av = 2v$ *and* $Aw = 4w$*?*
(c) Is there a hermitian matrix $E \in \mathcal{M}_{2,2}(\mathbb{C})$ *such that* $Av = 2v$ *and* $Aw = 4w$*?*

Ex. 0.3. *(a) Let* C *be an orthogonal matrix. Prove that* $\det C = +1$ *or* $\det C = -1$. *(Hint: use the fact that, for any square matrix,* $\det A = \det A^t$.
(b) Let $\mathcal{B} = \{A^1, A^2, A^3, A^4\}$ *be an orthogonal basis of* V_4 *such that* $\| A^1 \| = 3$, $\| A^2 \| = 5$, $\| A^3 \| = 2$, $\| A^4 \| = 2$. *Let* $A = (A^1\ A^2\ A^3\ A^4)$ *be the matrix whose columns are* A^1, A^2, A^3, A^4. *Compute* $|\det A|$.

Ex. 0.4. *Let* $\mathcal{B} = \{(1,2),(1,3)\}$. *Sia* $T : V_2 \to V_2$ *be the linear transformation such that* $m^{\mathcal{B}}_{\mathcal{B}}(T) = \begin{pmatrix} 9 & 13 \\ -6 & -9 \end{pmatrix}$. *Moreover, let* $S : \mathcal{V}_2 \to \mathcal{V}_2$ *be the linear transformation such that* $m^{\mathcal{B}}_{\mathcal{B}}(S) = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$. *Is* T *a symmetric linear transformation?. Is* S *a symmetric linear transformation?.*

Ex. 0.5. *Let* $v = (1,2,1)$. *Let us consider the function* $T : V_3 \to V_3$ *defined as follows:* $T(X) = X \times v$.
(a) Prove that T *is a linear transformation . (b) Is* T *symmetric?*
(b) Compute the matrix representing T *with respect to the canonical basis of* V_3 *and compare the result with the answer given to (a)*
(c) Find an orthonormal basis of V_3 $\mathcal{B} = \{v_1, v_2, v_3\}$ *with* v_1 *parallel to* v *and compute* $m^{\mathcal{B}}_{\mathcal{B}}(T)$.

Ex. 0.6. *Let* $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. *Let us consider the real quadratic form* $Q(X) = XAX^T$.

(a) Reduce Q to diagonal form by means of an orthonormal basis (that is: find an orthonormal basis $\mathcal{B}$ of V_4 and real numbers λ_i, $i = 1, 2, 3, 4$ such that $Q(X) = \sum_{i=1}^{4} \lambda_i {x'}^2$, where x'_i are the components of X with respect to the basis $\mathcal{B}$.
(b) Is Q a positive real quadratic form (this mean that $Q(X) \geq 0$ for all $X \in V_4$ and equality holds if and only if $X = O$).
(c) Compute the maximum and minimum value Q on the unite sphere S^3. Compute the points of maximum and the points of minimum.

Ex. 0.7. *Reduce the following real quadratic forms Q to diagonal form. Establish if they are positive, negative or what. Find explicitly (if any) non-zero vectors X such that $Q(X) > 0$ or $Q(X) < 0$ or $Q(X) = 0$. Find the maximum and minimum value of Q on the unit sphere and find the points of maximum and the points of minimum.*
(a) $Q(x, y) = x^2 - 3xy + y^2$.
(b) $Q(x, y, z) = -2x^2 - 5y^2 + 12yz + 7z^2$.
(c) $Q(x, y, z) = 4xy + 3y^2 + z^2$.
(d) $Q(x, y, z) = 25x^2 - 7y^2 + 48yz + 7z^2$.
(e) $Q(x, y, z) = \sqrt{2}(x^2 + y^2 + z^2) + 2x(y - z)$
(f) $Q(x, y, z) = 5x^2 - y^2 + z^2 + 4xy + 6xz$
(g) $Q(x, y, z) = 2(xy + yz + zx)$

Ex. 0.8. *Let $Q(x_1, x_2, x_3, x_4, x_5) = \sum_{1=1}^{5} x_i^2 + \sum_{i<j} x_i x_j$. Is Q positive?*

Ex. 0.9. *Reduce the equations of teh following conics to canonical form, find the symmetry axes, the center, the foci.. (everything in coordinates with respect to the canonical basis).*
(a) $5x^2 + 5y^2 - 6xy + 16\sqrt{2}x + 38 = 0$.
(b) $5x^2 - 8xy + 5y^2 + 18x - 18y + 9 = 0$
(c) $2x^2 - 2\sqrt{3}xy + 2x + 2\sqrt{3}y - 5 = 0$
(d) $x^2 + 2xy + y^2 + 2x - 2y = 0$
(e) $5x^2 + 8xy + 5y^2 - 18x - 18y + 9 = 0$