Linear Algebra and Geometry. 7th intermediate test, june 6, 2014.

Let $T : \mathcal{V}_3 \to \mathcal{V}_4$ be the linear transformation defined by

$$T((x, y, z, t)) = (x + y + z, x - 2y - z, x + y - z, -x + y).$$

(a) find dimension and a basis of $T(\mathcal{V}_3)$, $N(T)$, $N(T)^\perp$.
(b) For T varying on $\mathbf{R}$ let $\underline{v}_t = (1, 0, t, 2)$. Find all $t \in \mathbf{R}$ such that $\underline{v}_t \in \mathcal{V}_3$.
(c) Find two different vectors $\underline{u}, \underline{w} \in \mathcal{V}_3$ such that $T(\underline{u}) \neq T(\underline{w})$

Solution

Note that $T(X)^T = AX^T$, where $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. Gaussian elimination:

$$A \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix})$$

It is clear that $rk(A) = 3$. Therefore $rk(T) = \dim T(V_3) = \dim L(A^1, A^2, A^3) = 3$. This means that the three columns are independent, hence $\{A^1, A^1, A^3\}$ is a basis of $T(\mathcal{V}_3)$. Moreover, due to the (nullity)+(rank) Theorem, $\dim N(T) = 0$. Hence there no basis for $N(T)$.
Therefore $N(T)^\perp = \{O\}^\perp = \mathcal{V}_3$: its dimension is 3 and one can take the standard basis, for example. (In general, $N(T)^\perp$ is the linear span of the lines of A, and has dimension equal to $rk(A)$).
(b) $\underline{v}_t \in T(\mathcal{V}_3)$ if and only if the system $AX = \underline{v}_t$ has some solutions. Gaussian elimination of the augmented matrix:

$$A|\underline{v}_t \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 0 & -1 \\ 0 & 0 & 0 & t-1 \\ 0 & 2 & -1 & 3 \end{pmatrix})$$

It follows easily that the system has solution if and only if $t = 1$.
(c) Since $N(T) = \{O\}$, T is injective. therefore it is impossible to find different vectors $\underline{u}$ and $\underline{w}$ such that $T(\underline{u})] \neq T(\underline{w})$.