Linear Algebra and Geometry. Fourth intermediate test, april 11, 2014.

Let C be the ellipse whose vertices are $(4,3)$ and (14.3), and containing the point $(9,6)$.
(a) Write down the cartesian equation of C
(b) Find the points of C whose curvature is minimal, and compute the curvature at those points.

Solution. (a) Since the ellipse has a central symmetry, the center of symmetry has to be at midway between the two vertices. It has to belong to the line $y = 3$, and it must be (9.3). From what we know about ellipses, the line $y = 3$ is one of the two symmetry axes. Therefore the other symmetry axis is $x = 9$ and the equation must be of the form

$$\frac{(x-9)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$$

We have that a is the distance between the vertices and the center and therefore $a^2 = 25$ (this follows from the theory but also from an immediate calculation: just plug $(14,3)$ or $(4,3)$ in the equation). Plugging $(9,3)$ in the equation

$$\frac{(x-9)^2}{25} + \frac{(y-3)^2}{b^2} = 1$$

we find $b^2 = 9$. Therefore the equation is

$$\frac{(x-9)^2}{25} + \frac{(y-3)^2}{9} = 1$$

(b) Intuitively, the points of minimal curvature are the points on the line $x = 9$. To prove this, we take the following parametrization of C:

$$\mathbf{r}(t) = (9,3) + (5\cos t, 3\sin t)$$

Computing, we get

$$\mathbf{v}(t) = (-5\sin t, 3\cos t) \qquad v(t) = (25\sin^2 t + 9\cos^2 t)^{1/2} = (16\sin^2 t + 9)^{1/2}$$

$$\mathbf{a}(t) = (-5\cos t, -3\sin t)$$

Therefore $\| \mathbf{v}(t) \times \mathbf{a}(t) = (0,0,15\sin^2 t + 15\cos^2 t) = (0,0,15)$ and

$$\kappa(t) = \frac{\| \mathbf{v}(t) \times \mathbf{a}(t) \|}{v^3(t)} = \frac{15}{(16\sin^2 t + 9)^{3/2}}$$

Therefore the minimum is achieved when the denominator is maximal, hence for $|\sin t| = 1$, namely $t = \pi/2, (3\pi)/2)$. Therefore the points are $\mathbf{r}(\pi/2) = (9,3) + (0.3) = (9,6)$ and $(9,3) + (0,-3) = (9,3)$. At these points the curvature is

$$\frac{15}{25^{3/2}} = \frac{3}{\sqrt{5}}$$