LAG. 6th test (may 23 2014)

1. Let V be the space of real polynomials of degree ≤ 2.
let $U = \{ P(x) \in V \mid x P'(x) = P(x)+1 \}$ $V = \{ P(x) \in V \mid x P'(x) = P(x+1) \}$
$W = \{ P(x) \in V \mid (x+1) P'(x) = P(x) \}$.

(a) Which are linear subspaces of V? (Explain)
(b) Compute the dimension of those which are linear subspaces of V.

2. Let V be the space of all real polynomials. Which of the following formulas produces an inner product on V? (Explain ~~clearly~~ your answer)

(a) $(P,Q) = \int_0^1 P'(t) Q'(t)\,dt + \int_0^1 P''(t) Q''(t)\,dt$ | (b) $(P,Q) = \sum_{i=0}^{10} P(i) Q(i)$

(c) $(P,Q) = \left(\int_0^1 P(t)\,dt\right)\left(\int_0^1 Q(t)\,dt\right)$ | (d) $(P,Q) = \int_0^1 P(t) Q(t)\,dt + \int_0^1 P'(t) Q'(t)\,dt$

3. In V_4, with the usual dot product, let
$\underline{u} = (1,0,1,0)$ $\underline{v} = (1,0,0,1)$ $\underline{w} = (1,-1,1,0)$.
Find an orthogonal subset $S = \{\underline{a}, \underline{b}, \underline{c}\} \subset V_4$
such that $L(\underline{a}) = L(\underline{v})$, $L(\underline{a}, \underline{b}) = L(\underline{v}, \underline{w})$, $L(\underline{a}, \underline{b}, \underline{c}) = L(\underline{u}, \underline{v}, \underline{w})$.

SOLUTION

1 (a) – U is not a subspace ($0 \notin U$)

– V is a subspace: let P_1, P_2 such that
$x P_1'(x) = P_1(x+1)$ and $x P_2'(x) = P_2(x+1)$. Then

- $x(P_1+P_2)'(x) = (P_1+P_2)(x+1)$ because
  $x(P_1+P_2)'(x) = x P_1'(x) + x P_2'(x) = P_1(x+1) + P_2(x+1) = (P_1+P_2)(x+1)$
- if $\lambda \in \mathbb{R}$ $x(\lambda P_1)'(x) = (\lambda P_1)(x+1)$ because
  $x(\lambda P_1)'(x) = x \lambda P_1'(x) = \lambda x P_1'(x) = \lambda P_1(x+1) = (\lambda P_1)(x+1)$

W is a subspace:

let P_1, P_2 such that $(x+1)P_1'(x) = P_1(x)$ and
$(x+1)P_2'(x) = P_2(x)$

Then:

- $(x+1)(P_1+P_2)'(x) = (P_1+P_2)(x)$
- $(x+1)(\lambda P_1)'(x) = (\lambda P_1)(x)$

(The proof is completely similar to the previous one)

(b) $V = \{ a_0 + a_1 x + a_2 x^2 \,/\, \underset{a_1 x + 2a_2x^2}{\underset{\|}{x(a_1 + 2a_2 x)}} = \underset{= a_0 + a_1 + a_2 + (a_1+2a_2)x + a_2x^2}{\underset{\|}{a_0 + a_1(x+1) + a_2(x+1)^2}} \}$

That is
$$\begin{cases} a_0 + a_1 + a_2 = 0 \\ a_1 = a_1 + 2a_2 \\ a_2 = 2a_2 \end{cases}$$

Therefore $a_2 = 0$ $a_0 = -a_1$.

Hence V is the space of polynomials of form $P(x) = ax - a = a(x-1)$

Therefore $\dim V = 1$ (A basis is $\{x-1\}$).

$$W = \{ a_0 + a_1 x + a_2 x^2 \,/\, (x+1)(a_1 + 2a_2 x) = a_1 + (a_1 + 2a_2)x + 2a_2x^2 = a_0 + a_1 x + a_2 x^2 \}$$

Therefore
$$\begin{cases} a_1 = a_0 \\ a_1 + 2a_2 = a_1 \\ 2a_2 = a_2. \end{cases}$$

Hence $a_1 = a_0$ and $a_2 = 0$

Therefore W is the space of polynomials of form $a + ax = a(1+x)$

we have that $\dim W = 1$ (a basis is $\{x+1\}$).

2.1 (a) NO. Let $P(t) \equiv a$ $(a \neq 0)$ $(P,P) = 0$

(b) NO Let $P(t) = \cancel{t}(\cancel{t}-1)(\cancel{t}-2)\cdots(t-10)$. $(P,P) = 0$

(c) NO. Let $P(t) \not\equiv 0$ such that $\int_0^1 P(t)\,dt = 0$

(for example $P(t) = t - 1/2$). Then $(P,P) = 0$

(d) YES. All the axioms are satisfied:

- $(P,Q) = (Q,P)$ (Obvious)

- $(P_1 + P_2, Q) = (P_1, Q) + (P_2, Q)$ Indeed:

$$(P_1+P_2, Q) = \int_0^1 (P_1(t) + P_2(t))Q(t)\,dt + \int_0^1 (P_1'(t) + P_2'(t))Q'(t)\,dt =$$

$$= \int_0^1 P_1(t)Q(t)\,dt + \int_0^1 P_1'(t)Q'(t)\,dt + \int_0^1 P_2(t)Q(t)\,dt + \int_0^1 P_2'(t)Q'(t)\,dt$$

$$= (P_1, Q) + (P_2, Q)$$

- $(\lambda P, Q) = \lambda(P,Q)$ Indeed.

$$(\lambda P, Q) = \int_0^1 \lambda P(t)Q(t)\,dt + \int_0^1 \lambda P'(t)Q'(t)\,dt =$$

$$= \lambda\left(\int_0^1 P(t)Q(t)\,dt + \int_0^1 P'(t)Q'(t)\,dt\right) =$$

$$= \lambda(P,Q).$$

- $(P,P) \geq 0$ $\forall P$ and: $(P,P) = 0 \iff P = 0$.

Indeed

$$(P,P) = \int_0^1 P^2(t)\,dt + \int_0^1 (P'(t))^2\,dt \geq 0$$

Moreover $(P,P) = 0 \Longrightarrow \int_0^1 P^2(t)\,dt = 0 \Rightarrow P^2(t) \equiv 0 \Rightarrow P(t) \equiv 0$

3| We take $\underline{a} = \underline{v} = (1,0,0,1)$

$$\underline{b} = \underline{w} - \left(\frac{\underline{a}\cdot\underline{w}}{\underline{a}\cdot\underline{a}}\right)\underline{a} = (1,-1,1,0) - \tfrac{1}{2}(1,0,0,1) = \left(\tfrac{1}{2}, -1, 1, -\tfrac{1}{2}\right)$$

$$\underline{c} = \underline{u} - \left(\frac{\underline{a}\cdot\underline{u}}{\underline{a}\cdot\underline{a}}\right)\underline{a} - \left(\frac{\underline{b}\cdot\underline{u}}{\underline{b}\cdot\underline{b}}\right)\underline{b} = (1,0,1,0) - \frac{1}{2}(1,0,0,1) - \frac{3/2}{5/2}\left(\tfrac{1}{2},-1,1,-\tfrac{1}{2}\right) =$$

$$= \left(\tfrac{1}{2},0,1,-\tfrac{1}{2}\right) - \frac{3}{5}\left(\tfrac{1}{2},-1,1,-\tfrac{1}{2}\right) = \left(\tfrac{1}{5},\tfrac{3}{5},\tfrac{2}{5},-\tfrac{1}{5}\right)$$

In conclusion: $\{\underline{a},\underline{b},\underline{c}\} = \left\{(1,0,0,1), \left(\tfrac{1}{2},-1,1,-\tfrac{1}{2}\right), \left(\tfrac{1}{5},\tfrac{3}{5},\tfrac{2}{5},-\tfrac{1}{5}\right)\right\}$.