Linear Algebra and Geometry. Written test of february 10, 2014.
(Note: answers without adequate justification and proofs – especially in Exercises 2, 3 and 5 – will not be evaluated)

1. Let $\mathbf{u} = (1, 1, -1)$ and $\mathbf{v} = (1, 0, -1)$. (a) Find an orthogonal basis $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ of $\mathcal{V}_3$ such that (a) $\mathbf{w_1}$ is perpendicular to $L(\mathbf{u}, \mathbf{v})$ and $\mathbf{w_2} = \mathbf{u}$.
(b) Find an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ of $\mathcal{V}_3$ such that $L(\mathbf{v_1}, \mathbf{v_2}) = L(\mathbf{u}, \mathbf{v})$ and the angle between $\mathbf{v_1}$ and $\mathbf{u}$ is $\pi/3$.

2. Consider the conic section passing trough the point $(-3, -5)$ and having the lines $y = 3x + 1$ and $y = -3x - 5$ as asymptotes. Find its cartesian equation.

3. A point moves in $\mathcal{V}_2$ according to the equation $\mathbf{r}(t) = ((x(t), y(t))$. The acceleration $\mathbf{a}(t)$ is directed toward the origin and its norm is four times the norm of $\mathbf{r}(t)$. At time $t = 0$ the initial position is $\mathbf{r}(0) = (4, 0)$ and the initial velocity is $\mathbf{v}(0) = (0, 6)$.
(a) Determine explicitly $x(t)$ and $y(t)$ in function of t. (b) Find the cartesian equation of the path of the point, sketch it and indicate the direction of motion.

4. For t varying in $\mathbf{R}$ let $A_t = \begin{pmatrix} t-1 & 2t & 4t & 1 \\ 2t-2 & 1 & 2 & 3 \\ t-1 & -t & -2t & 1 \\ 0 & 1 & 5 & 1 \end{pmatrix}$.
(a) Compute $\det A_t$ as a function of t; (b) Find all $t \in \mathbf{R}$ such that A_t is not invertible and, for such values of t, compute the rank of A_t.

5. Find all real symmetric matrices $A \in \mathcal{M}_3(\mathbf{R})$ satisfying the following conditions
(a) $A \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$; (b) $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is an eigenvector of A; (c) $Tr(A) = 0$; (d) $\det A = -30$.
(recall that $Tr(A)$, the *trace* of A is defined as $Tr(A) = a_{11} + a_{22} + a_{33}$)