Linear Algebra and Geometry. First intermediate test, march 14 2014.

First version. Let $A = (1, 1, -1)$, $B = (5, 2, -4)$ and $C = (2, 2, 3)$. Find all vectors D of the form $xA + yB$ which are orthogonal to C.

Solution. $xA + yB = (x + 5y, x + 2y, -x - 4y)$. Therefore

$(xA+yB)\cdot C = (x+5y, x+2y, -x-4y)\cdot(2, 2, 3) = 2x+10y+2x+4y-3x-12y = x+2y$

Therefore $(xA+yB)\cdot C = 0$ if and only if $x = -2y$. This means that the vectors $xA+yB$ which are orthogonal to C are those of the form

$$-2yA + yB = y(-2A + B) = y(3, 0, -2)$$

Now

$$\| \ y(3, 0, -2) = |y| \ \| \ (3, 0, -2) \ \|= |y|\sqrt{13}.$$

Therefore the norm is 10 if and only if $|y|\sqrt{13} = 10$, that is $y = 10/\sqrt{13}$ and $y = -10/\sqrt{13}$. The final answer is: there are two vectors as required:, namely

$$D_1 = \frac{10}{\sqrt{13}}(3, 0, -2) \qquad \text{and} \qquad D_2 = \frac{-10}{\sqrt{13}}(3, 0, -2).$$

Second version. Let $A = (2, -3, 2)$, $B = (1, 1, 1)$ and $C = (0, -1, -3)$. Find all vectors D of the form $xB + yC$ which are orthogonal to A.

Solution. $xB + yC = (x, x - y, x - 3y)$. Therefore

$(xB + yC)\cdot A = (x + 5y, x + 2y, -x - 4y)\cdot(2, -3, 2) = 2x - 3(x - y) + 2(x - 3y) = x - 3y$

Therefore $(xB + yC)\cdot A = 0$ if and only if $x = 3y$. This means that the vectors $xA + yB$ which are orthogonal to C are those of the form

$$3yB + yC = y(3B + C) = y(3, 2, 0)$$

Now

$$\| \ y(3, 2, 0) = |y| \ \| \ (3, 2, 0) \ \|= |y|\sqrt{13}.$$

Therefore the norm is 10 if and only if $|y|\sqrt{13} = 10$, that is $y = 10/\sqrt{13}$ and $y = -10/\sqrt{13}$. The final answer is: there are two vectors as required, namely

$$D_1 = \frac{10}{\sqrt{13}}(3, 2, 0) \qquad \text{and} \qquad D_2 = \frac{-10}{\sqrt{13}}(3, 2, 0).$$