**L.A.G. . Solution of some exercises on the textbook, I**

## Section 14.13

**Exercise n. 11 Sect. 14.13. Vol. I.** First we find a parametrization of the curve. Clearly $x(t) = t^2$, $y(t) = t^3$ works. Hence $\mathbf{r}(t) = (t^2, t^3)$. Therefore, with this parametrization, $\mathbf{v}(t) = (2t, 3t^2)$ and $v(t) = \sqrt{4t^2 + 9t^4} = |t|\sqrt{4 + 9t^2}$. A primitive of $t\sqrt{4 + 9t^2}$ is $(1/27)(4 + 9t^2)^{3/2}$.

Note that, in the parametrization above, $(1, -1)$ corresponds to $t = -1$ and $(1, 1)$ corresponds to $t = 1$. In conclusion, the lenght of the arc joining the points $(1, -1)$ and $(1, 1)$ is

$$\int_{-1}^{0} -t\sqrt{4 + 9t^2}dt + \int_{0}^{1} t\sqrt{4 + 9t^2}dt = \big[(1/27)(4 + 9t^2)^{3/2}\big]_0^{-1} + \big[(1/27)(4 + 9t^2)^{3/2}\big]_0^1 =$$

$$= \frac{1}{27}(13\sqrt{13} - 8) + (13\sqrt{13} - 8) = \frac{26\sqrt{13} - 16}{27}$$

**Exercise n. 12 Sect. 14.13. Vol. I.** Parametrization of the circle of radius 1 and center $\mathbf{0}$: $\mathbf{r}(\theta) = (\cos\theta, \sin\theta)$. The speed is constant, equal to 1. Length of the arc: $\int_{\theta_1}^{\theta^2} 1 d\theta = \theta_2 - \theta_1$. Area of the sector: $(1/2)\int_{\theta_2}^{\theta^2} 1^2 d\theta = (1/2)(\theta_2 - \theta_1)$.

**Exercise n. 13 Sect. 14.13. Vol. I.** (a) $\mathbf{r}(x) = (x, e^x)$, $\mathbf{v}(x) = (1, e^x)$, $v(x) = \sqrt{1 + e^{2x}}$. Length: $\int_0^1 \sqrt{1 + e^{2x}}dx$.
(b) $\mathbf{r}(t) = (t + \log t, t - \log t)$, $\mathbf{v}(t) = (1 + (1/t), 1 - (1/t))$, $v(t) = \sqrt{(1 + (1/t))^2 + (1 - (1/t))^2} = \frac{\sqrt{2}\sqrt{1+t^2}}{t}$.
Length: $\sqrt{2}\int_1^e \frac{\sqrt{1+t^2}}{t}dt$.
The integral in (a) is equal to the integral in (b) via the substitution $t = e^x$.

**Exercise n. 19 Sect. 14.13. Vol. I.** $\mathbf{r}(t) = tA + t^2B + 2(\frac{2}{3}t)^{3/2}A \times B$. $\mathbf{v}(t) = A + 2tB + 3(\frac{2}{3}t)^{1/2}\frac{2}{3}A \times B = A + 2tB + 2(\frac{2}{3}t)^{1/2}A \times B$.
$v(t) = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)} = \sqrt{\| A \|^2 + 4t^2 \| B \|^2 + 4(\frac{2}{3}t) \| A \times B \|^2 + 4tA \cdot B} = \sqrt{1 + 4t^2 + \frac{8}{3}\sin^2\frac{\pi}{3}t + 4\cos^2\frac{\pi}{3}t} =$
$= \sqrt{1 + 4t^2 + 4t} = |1 + 2t|$.
We are lloking for the $t > 0$ such that $\int_0^t (1 + 2u)du = 12$, that is: $12 = \big[\frac{(1+2u)^2}{2}\big]_0^t = \frac{(1+2t)^2}{2} - \frac{1}{2}$. Computing one gets $(1 + 2t)^2 - 1 = 24$. The solution is $t = 3$.

## Section 14.15

**Exercise n. 7 Sect. 14.15. Vol. I.** Since the speed is constant we have that $\| \mathbf{a} \| = v \| T' \|$. On the other hand $\kappa = \| T' \| /v = \| \mathbf{a} \| /v^2$.

**Exercise n. 11 Sect. 14.15. Vol. I.** We know that $\kappa(t) = |\alpha'(t)|/v(t)$ (as in Example 2 of §14.14 $\kappa(t) = |d\alpha/ds| = |(d\alpha/dt)(dt/ds)| = |(d\alpha/dt)|/(ds/dt)| = |\alpha'/v|$). In our case: $2t = |\alpha'(t)|/5$. The hypothesis tells that $\alpha(0) = \pi/2$ and then $\alpha(t) \leq \pi/2$. Hence $\alpha'(t) = -10t$ (it cannot be $\alpha'(t) = 10t$ because, if this was the case, $\alpha(t)$ would be bigger than $\pi/2$ for $t > 0$. Therefore

$$\alpha(t) = -5t^2 + c$$

where $c$ is a constant. Since we know that $\alpha(0) = \pi/2$ we have that $c = \pi/2$. In conclusion $\alpha(t) = -5t^2 + \pi/2$. Therefore

$$\mathbf{v}(t) = v(\cos\alpha(t), \sin\alpha(t)) = 5(\cos(-5t^2 + \frac{\pi}{2}), \sin(-5t^2 + \frac{\pi}{2}))$$

## Section 14.19

**Exercise n. 6 Sect. 14.19. Vol. I.** (a) For $c = 0$ it is a circle of radius one. Otherwise it is a spiral, going from the interior to the exterior for $c = 1$, and from the exterior from the interior for $c = -1$.
(b) Calculation of the speed in polar coordinates: since in this exercise the parameter is the angle $\theta$ the formulas simplify as in the last lines of §15.16. Therefore $v(\theta) = \sqrt{(\rho')^2 + \rho^2} = \sqrt{c^2 e^{2c\theta} + e^{2c\theta}} = e^{c\theta}\sqrt{c^2+1}$.

Hence $L(c) = \int_0^{2\pi} v(\theta)d\theta = \sqrt{c^2+1}\int_0^{2\pi} e^{c\theta}d\theta = \frac{\sqrt{c^2+1}}{c}\left[e^{c\theta}\right]_0^{2\pi} = \frac{\sqrt{c^2+1}}{c}(e^{2c\pi}-1)$.

$R(c) = \frac{1}{2}\int_0^{2\pi}\rho^2(\theta)d\theta = \int_0^{2\pi} e^{2c\theta} = \frac{1}{2c}(e^{4c\pi}-1)$.

**Exercise n. 14 Sect. 14.19. Vol. I.** We assume that the parameter is the angle $\theta$. Hence (formulas at the end of §15.16) $\mathbf{v} = \frac{d\rho}{d\theta}\mathbf{u}_\rho + \rho\mathbf{u}_\theta$. Since $\{\mathbf{u}_\rho, \mathbf{u}_\theta\}$ is an orthonomal basis (varying with $\theta$) we have that , since $\mathbf{r} = \rho\mathbf{u}_\rho$,

$$\rho v\cos\phi = \mathbf{r}\cdot\mathbf{v} = \rho\frac{d\rho}{d\theta}$$

and

$$\rho v\sin\phi = \parallel \mathbf{r}\times\mathbf{v} \parallel = \parallel \rho\mathbf{u}_\rho \times \rho\mathbf{u}_\theta \parallel = \rho^2$$

**Exercise n. 15 Sect. 14.19. Vol. I.** This exercise is closely related to the previous two. Indeed, note that if we have a plane motion expressed by the polar equation $\rho = Ke^{c\theta}$ (see Exercise n.6 above) then the angle $\phi = \phi(\theta)$ between $\mathbf{r}$ and $\mathbf{v}$ is *constant* (note: this is a property of the *curve*, and not of the parametrization). This is seen, for example, as follows: from the previous exercise $\tan\phi = \rho'/\rho \equiv c$.

Also the converse is true: in fact, if $\tan\phi \equiv c$, again from the previous exercise, $\rho'/\rho \equiv c$ and, integrating, $\log\rho = c\theta + d$. Exponentiating: $\rho = Ke^{c\theta}$, with $K = e^d$.

Thus we have seen that:
*a regular plane curve has the property that the angle between the position and velocity vectors is constant if and only if its polar equation is of the form $\rho = Ke^{c\theta}$, for $K, c \in \mathbf{R}$* (such curves are called *logarithmic spirals*).

Having said that, the solution of this exercise is easy: we take the target as origin. The "direction in actual flight" at a given time is the direction of the velocity vector, so the hypothesis is that the angle between $\mathbf{v}$ and $-\mathbf{r}$ is constant equal to $\alpha$. Hence the angle beween $\mathbf{v}$ and $\mathbf{r}$ is constant equal to $\pi - \alpha$. Therefore the path is a logarithmic spiral of polar equation $\rho = e^{c\theta}$, with $c = \tan(\pi - \alpha)$. Hence it is a circle for $\alpha = \pi/2$, the distance from the origin goes to $\infty$ when $t \to \infty$ if $\alpha > \pi/2$ (that is $c > 0$), and the distance from the origin goes to 0 for $t \to \infty$ if $\alpha < \pi/2$ (that is $c < 0$). In this last case one could say that "yes, the missile will reach the target" in the sense that it will get closer and closer to the target.

**Exercise n. 5 Sect. 14.19. Vol. I.** For $\theta = 0$ we have that $\rho = 8$, when $\theta$ increses in $[0, \pi]$ $\rho$ decreases. For $\theta = \pi$ we have $\rho = 0$. The curve is symmetric with respect to the $x$ axis, so the piece of curve corresponding to $[\pi, 2\pi]$ is the curve – under the $x$-axis – symmetric to the previous one. Note that the curve is not *regular*, that is to say $\mathbf{v} = \mathbf{0}$ for $\theta = \pi$ (exercise!). The overall picture is a "horizontal" heart (whence the name *cardioid*).

Length: $v(\theta) = \sqrt{(\rho')^2 + \rho^2} = \sqrt{(4\sin\theta)^2 + (4(1+\cos\theta))^2} = 4\sqrt{2+2\cos\theta} = 4\sqrt{2}\sqrt{1+\cos\theta} =$
$= 4\sqrt{2}\sqrt{1+\cos^2(\theta/2) - \sin^2(\theta/2)} = 8|\cos(\theta/2)|$
Therefore the length is $\int_0^{2\pi} v(\theta)d\theta = 8(\int_0^{\pi}\cos(\theta/2) - \int_\pi^{2\pi}\cos(\theta/2)d\theta) = 32$.

**Exercise n. 12-13 Sect. 14.19. Vol. I.** Curvature in polar coordinates (if the parameter is the angle $\theta$).
$\mathbf{v} = \rho'\mathbf{u}_\rho + \rho\mathbf{u}_\theta$. $\quad v = \sqrt{(\rho')^2 + \rho^2}$
$\mathbf{a} = (\rho'' - \rho)\mathbf{u}_\rho + 2\rho'\mathbf{u}_\theta$.
$\mathbf{v}\times\mathbf{a} = 2(\rho')^2\mathbf{u}_\rho \times \mathbf{u}_\theta + \rho(\rho'' - \rho)\mathbf{u}_\theta \times \mathbf{u}_\rho = 2(\rho')^2 - \rho(\rho'' - \rho)\mathbf{u}_\rho \times \mathbf{u}_\theta$
$\kappa = \parallel \mathbf{v}\times\mathbf{a} \parallel /(v^3) = |2(\rho')^2 - \rho(\rho'' - \rho)|/((\rho')^2 + \rho^2)^{3/2}$

Radius of curvature: $= 1/\kappa$.
At this point the Exercises 13 are easy.

**Exercise n. 3 Sect. 14.19. Vol. I.** We recall that *cylindrical coordinates* in $\mathcal{V}_3$ are coordinates $(\rho, \theta, z)$ such that $(\rho, \theta)$ are the usual polar coordinates of the point $(x, y)$ in the $(x, y)$-plane.
(a) Since $\rho = \sin\theta$ we have that $(x, y) = \rho(\cos\theta, \sin\theta) = \sin\theta(\cos\theta, \sin\theta)$. Therefore $x^2 + (y - 1/2)^2 = \sin^2\theta\cos^2\theta + (\cos^2\theta - (1/2))^2 = \sin^2\theta\cos^2\theta + \cos^4\theta - \cos^2\theta + (1/4) = (1 - \cos^2\theta)\cos^2\theta + \cos^4\theta - \cos^2\theta + (1/4) = 1/4$. (Note that this means that if $(x, y, z)$ belongs to the curve of the exercise then $(x, y)$ belongls to the circle of radius $1/2$ and center $(0, 1/2)$ in the $(x, y)$-plane).
(b) In cylindrical coordinates $\mathbf{r} = \rho\mathbf{u}_\rho + z\mathbf{k}$. Therefore $\mathbf{v} = \rho'\mathbf{u}_\rho + \rho\theta'\mathbf{u}_\theta + z'\mathbf{k}$. Since in this case the parameter is $\theta$ itself (that is to say: $t = \theta$) this simplifies as $\mathbf{v} = \rho'\mathbf{u}_\rho + \rho\mathbf{u}_\theta + z'\mathbf{k}$. Therefore the angle between $\mathbf{v}$ and the $z$-axis is

$$\arccos\frac{z'}{v}$$

Now (in the hypothesis $\theta = t$) we have $v = \sqrt{(\rho')^2 + \rho^2 + (z')^2}$. In our case: $v = \sqrt{\cos^2\theta + \sin^2\theta + (z')^2} = = \sqrt{1 + (z')^2}$. In conclusion the angle between $\mathbf{v}$ and the $z$-axis is

$$\arccos\frac{z'}{\sqrt{1 + (z')^2}}$$

In our case:

$$z = z(\theta) = \log(\frac{1}{\cos\theta}) \qquad \text{Hence} \qquad z' = \cos\theta\frac{-(-\sin\theta)}{\cos^2\theta} = \tan\theta$$

Therefore the required angle is

$$\arccos(\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}) = \arccos(\frac{\sin\theta}{\cos\theta}\frac{1}{\sqrt{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}}}) = \arccos(\frac{\sin\theta\cos\theta}{\cos\theta}) = \arccos(\sin\theta) = \frac{\pi}{2} - \theta$$